Introduction to Symplectic Topology

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June 17, 2026

What do we mean when we say "symplectic geometry", or what does a random person mean when he says he is a "symplectic geometor"? This name has been used to refer to many completely different areas of research in the mathematics community, though there does exist a group of people who claim they are really working on symplectic geometry, and didn't think other stuffs can be called this name at all.

This course, under the name "Introduction to Symplectic Topology", aims to clarify what we really mean by "Symplectic Geometry", or "Symplectic Topology", and to give audiences a sense of what symplectic geometors regard as an interesting symplectic-geometric problem.

During this fifteen-week (and a total of thrity lectures) course, we will first go through the main storyline of symplectic geometry, and then turn to three selected topics by the lecturer about some of the side stories. Unlike many other areas of research in mathematics, the "main storyline" of symplectic geometry ended quickly, and the current researches are like tangling vines growing out of the trunk. Any such finite selection of topics will be therefore always biased, and we hope the three topics selected can be of any help to the audiences.

Due to limitations of time, we are unfortunately unable to provide completely rigorous proofs of everything, leaving only sketches and ideas of proofs, where audiences are expected to fill in the remaining details themselves.

Oral Exam Patterns. During the lecture, there will be constantly practice questions popped out to the audiences, and they will be the final list of questions that will be (selectively) tested in the oral exam at the end of the course. There'll be two types of questions: basic ones, which consists of 50% of the total questions asked, which examinees have to answer all the questions asked; topic-based ones, which will be marked with a star ($\star$) and consists of the remaining 50% of the total questions asked, where examinees can choose which questions to answer before the exam. They will not be covered in the lecture, but references will be provided and the examinees are expected to learn the references and answer the questions correctly. Solutions to questions of this type will already be inside the references provided, and examinees only need to understand the given literatures.

Contents

1. From Classical Mechanics

Classical mechanics focuses on motions of particles, where by Newton's second law, the accerleration of the motion of a particle is determined by the mass of the particle and **forces** imposing on it.

(1a) Newton's second law. Before we move forward, let's make some conventions. Particles will always move on a smooth manifold M of dimension m, which is called the **configuration space** of the mechanical system.

1.1 Definition. Let M be a smooth manifold. A **Riemannian metric** on M is a symmetric tensor $g\colon TM \otimes TM \to \mathbb{R}$ such that g_q is positive definite for all $p \in M$.

Riemannian metric in classical mechanics language is the choice of a metric, i.e. units associated to relative positions and vectors, like meters per second (m/s) or Newton ($N = kg \cdot m/s^2$). A Riemannian metric g on M enables us to define the **length** of a tangent vector $v \in T_qM$ as

$$|v| := \sqrt{g_q(v, v)},$$

and similarly the inner product between vectors v and w on T_qM as

$$\langle v, w\rangle := g_q(v, w).$$

Suppose that $\gamma\colon [a, b] \to M$ is the trajectory of a particle moving on the configuration space M, then its **velocity** is the tangent map $T\gamma\colon T[a, b] \to TM$, or equivalently, section of the vector bundle

$$\gamma^*TM \otimes T[a, b] \cong \gamma^*TM \to [a, b]$$

given by $t \mapsto (\gamma(t), \dot\gamma(t))$. The Riemannian metric g induces a canonical decomposition $TTM \cong TM \oplus TM$, so that $T\gamma$ induces a further tangent map $T^2\gamma\colon T^2[a, b] \to T^2M$ which we write as $T^2\gamma(t) = (\gamma(t), \dot\gamma(t), \dot\gamma(t), \ddot\gamma(t))$. The vector $\ddot\gamma(t) \in T_qM$ is called the **acceleration** of the movement of particles.

Newton's first law of motion states that if there're no forces imposing on a particle, then $\ddot\gamma(t) \equiv 0$, i.e. the particle will move with constant velocity on M, and with the Riemannian metric g imposed, we know that this has to be the geodesic of M with the given initial velocity.

Newton's second law proposes that the force is the change of motions, and should hence be proportional to the acceleration and the mass of the particle. Assume that the particle has mass 1, and let $(\gamma, \tau)\colon (a, b) \to TM$ be any curve where particles are moving on, then Newton's second law can be reformulated into the following:

1.2 Proposition (Newton's Second Law).

$$\begin{cases} \dot\gamma(t) = \tau(t) \\ \dot\tau(t) = \dfrac{1}{m}F(\gamma(t), \tau(t)) \end{cases}$$

where $F\colon TM \to TTM \to TM$ is a section with respect to the projection $TTM \cong TM \oplus TM \to TM$ to the orthogonal complement of the horizontal distribution $T_qM \hookrightarrow T^hM \to TM$ over TM. Recall that for ordinary differential equations $\dot x(t) = f(x(t))$ with f smooth, the existence and uniqueness of solutions are always guaranteed:

1.3 Theorem ([Lee13], page 664). Suppose $U \subseteq \mathbb{R}^n$ open and $f\colon U \to \mathbb{R}^n$ a smooth vector-valued function. For the initial value problem

$$\dot{x}(t) = f(x(t)),$$
$$x(t_0) = x_0$$

for arbitrary $t_0 \in \mathbb{R}$ and $x_0 \in U$, there exists an open interval $I \subseteq \mathbb{R}$ containing t_0 and an open subset $U_0 \subseteq U$ containing x_0 such that for each $x \in U_0$, there is a C^1-map $x\colon I \to U$ solving this initial value problem. Moreover, any two such solutions agree on their common domain.

If $\theta\colon I \times U_0 \to U$ is the map $\theta(t, y) = x(t)$, where $x(t)$ solves the differential equation with initial condition $x(0) = y$, then θ is smooth.

In other words, any motion of particles, given the force function $F\colon TM \to \mathbb{R}^m$, is completely determined by the position it starts with and the instantaneous velocity at that position.

1.4 Example (Harmonic Oscillator). If the force is given by Hooke's law, $F(x) = -kx$, where k is a positive constant, then Newton's law can be written as

$$\begin{cases} \dot{\gamma}(t) = \tau(t) \\ \dot{\tau}(t) = -\dfrac{k}{m}\gamma(t) \end{cases}$$

On the space $TM \cong \mathbb{R}^2$. The general solution of this system is $x(t) = a\cos(\omega t) + b\sin(\omega t)$, where $\omega := \sqrt{k/m}$ is the frequency of oscillation.

Physically, this system can be described as a mass on a spring, where the force is proportional to the distance x that the string is stretched from its equilibrium position. The minus sign in $-kx$ indicates that the force pulls the oscillator back toward equilibrium.

1.5 Example (The Kepler Problem). The classical Kepler problem describes the motion of a planet around a star under the influence of gravity. The configuration space is $M = \mathbb{R}^3 \setminus \{0\}$, and the force function is given by Newton's law of universal gravitation:

$$F(q) = -\frac{GM}{|q|^3} q,$$

where G is the gravitational constant, M is the mass of the star, and q is the position vector of the planet relative to the star. The negative sign indicates that the force is attractive, pulling the planet towards the star. The solutions to this system are conic sections (ellipses, parabolas, or hyperbolas) depending on the total energy of the system. However, this system is not easily solved from directly from the ODEs, and we will revisit this example later when we introduce some geometric techniques.

(1b) Hamilton formalism. Determination of the force function $F\colon TM \to \mathbb{R}^m$ on the whole tangent bundle TM is not practical in general, and we hope to have a simpler and more handable way to determine the mechanical system. Let

$$L = \frac{1}{2} m|v|^2 - V(q, v)$$

be the **action** of the mechanical system, where $V\colon TM \to \mathbb{R}$ is called the **potential energy** of the system. Physicists observe that

1.6 Principle (The Principle of Least Action). Motions of particles should minimize the action $L\colon TM \to \mathbb{R}$, i.e. if $x = (q, v)\colon (a, b) \to TM$ is the curve of motion of a particle, then for any other curve $y = (q', v')\colon (a, b) \to TM$ with the same endpoints as x, we have

$$\int_a^b L(x(t))\,dt \leq \int_a^b L(y(t))\,dt.$$

The principle of least action determines for us the motion of particles, given by the following **Euler-Lagrange equation**

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial v_i} = 0, \quad i = 1, \cdots, m. \tag{1}$$

The **Hamilton formalism** is an equivalent way to write down the system of motion from not the action functional L, but from the energy function E, where

$$E(q(t), v(t)) = \frac{1}{2}mv(t)^2 + V(q(t), v(t)).$$

This function is conserved along the trajectories of the system, i.e. assume that $x = (q, v)\colon (a, b) \to TM$ is the curve of motion of a particle, we must have

$$\frac{d}{dt}E(x(t)) = \left\langle \frac{\partial E}{\partial q}, \dot{q}(t) \right\rangle + \left\langle \frac{\partial E}{\partial v}, \dot{v}(t) \right\rangle = 0.$$

In fact, similar to the principle of least action, the energy function will completely determine the motion of particles, if we replace velocity coordinates v by momentum coordinates $p = mv$. Moments should be regarded as the dual of velocities, and the energy function

$$E(q, p) = \frac{1}{2m}p^2 + V(q)\colon T^*M \to \mathbb{R}$$

can be regarded as a function defined on the **phase space** T^*M of the system, which geometrically is the cotangent bundle of the configuration space M. The system of motions will be the following systems of ordinary differential equations

$$\begin{cases} \dot{q}(t) = \frac{\partial E}{\partial p}(q(t), p(t)) \\ \dot{p}(t) = -\frac{\partial E}{\partial q}(q(t), p(t)) \end{cases} \tag{2}$$

This is called the **Hamilton's system**, and we can interchange between these two formalisms via **Legendre transform**.

(1c) Geometric structures of mechanics. The reason that we turn our attentions to the Hamilton formalism and phase spaces is that the system of motions can be described simply by a smooth function $H\colon T^*M \to \mathbb{R}$, where the system of motions can be derived from an intrinsic geometric structure of the phase space T^*M.

(1c-i) Symplectic manifolds. We start with some slightly more general definitions, introduced originally by Arnold [Ad78] to describe the intrinsic geometric structure of classical mechanics.

1.7 Definition. For a given vector space V, a **symplectic structure** on V is a non-degenerate skew-symmetric bilinear form $\Omega\colon \wedge^2 V \to \mathbb{R}$. Let M be a smooth manifold. A **symplectic form** on M is a closed and non-degenerate 2-form $\omega \in C^\infty(M, \wedge^2 T^*M)$, i.e. for each $p \in M$, the bilinear form $\omega(p)\colon T_pM \times T_pM \to \mathbb{R}$ is non-degenerate. A **symplectic manifold** is a pair (M, ω) where M is a smooth manifold and ω is a symplectic form on M.

It's immediate from the definition that a symplectic manifold has to be even-dimensional. Non-degeneracy of ω also induces an isomorphism $\omega\colon TM \to T^*M$, which provides a bijection between vector fields $X \in C^\infty(M, TM)$ and 1-forms $\alpha \in C^\infty(M, T^*M)$ on M.

1.8 Lemma. A vector field X on (M, ω) is called **symplectic** if $\mathcal{L}_X\omega = 0$; a vector field X is symplectic if and only if the symplectic dual $\alpha(-) := \omega(-, X)$ is a closed 1-form; a vector field X is called **hamiltonian** if the symplectic dual α is an exact 1-form, i.e. there exists a smooth function $H\colon M \to \mathbb{R}$ such that $\alpha = dH$. In this case, we call H the **hamiltonian** of the hamiltonian vector field X.

Given a smooth function $H\colon M \to \mathbb{R}$, the vector field X_H dual to dH is called the **hamiltonian vector field** of H.

Proof. By Cartan's magical formula we have

$$\mathcal{L}_X\omega = d(X \lrcorner \omega) + X \lrcorner d\omega.$$

As ω is closed, $\mathcal{L}_X\omega = d(X \lrcorner \omega) = d\alpha$ is zero iff $d\alpha = 0$, i.e. α is a closed 1-form. □

Note that the flow ϕ_H^t of a hamiltonian vector field X_H provides a smooth family of symplectomorphisms $\phi_H^t\colon M \to M$ for $t \in \mathbb{R}$.

1.9 Definition. We write $\mathrm{Symp}(M, \omega)$ for the group of symplectomorphisms of (M, ω), and $\mathrm{Ham}(M, \omega)$ for symplectomorphisms of M which are time 1 flow of some one-parameter family of hamiltonian vector fields $\{X_{H_t}\}_{t\in[0,1]}$ for some time-dependent Hamiltonians $\{H_t\}_{0\le t\le 1}$.

1.10 Remark. Though we only verify that for a single Hamiltonian $H\colon M \to \mathbb{R}$, the hamiltonian vector field X_H integrates to a smooth family of symplectomorphisms $\phi_H^t\colon M \to M$, this is also true for time-dependent hamiltonians $H_t\colon M \times [0, 1] \to \mathbb{R}$, where the flow $\phi_{H_t}^t\colon M \to M$ is defined as the family of diffeomorphisms such that $\dfrac{d}{dt}\phi_{H_t}^t = X_{H_t}$ for each t.

$\mathrm{Ham}(M, \omega)$ is not only a subset of $\mathrm{Symp}(M, \omega)$ but actually a normal subgroup, as we will prove in the next Proposition.

1.11 Proposition. Let ϕ, ψ be Hamiltonian diffeomorphisms generated by time-dependent Hamiltonians H_t, G_t, then

1. The composition $\phi \circ \psi$ is a Hamiltonian diffeomorphism generated by $H_t + G_t \circ \phi^{-1}$.
2. ϕ^{-1} is a Hamiltonian diffeomorphism generated by $-H_t \circ \phi$.
3. For any symplectomorphism $\chi \in \mathrm{Symp}(M, \omega)$, the conjugation $\chi^{-1} \circ \phi \circ \chi$ is a Hamiltonian diffeomorphism generated by $H_t \circ \chi$.

Proof. We only prove the first statement as a proof pattern. The reader can verify the remainings themselves. We write ϕ^t and ψ^t for the Hamiltonian flows generated by the given Hamiltonians, then by taking derivatives, we have

$$\left.\frac{d}{dt}\right|_t (\phi^t \circ \psi^t) = X_{H_t} \circ \phi^t \circ \psi^t + (\phi^t)_* X_{G_t} \circ \psi^t = X_{H_t} \circ \phi^t \circ \psi^t + X_{G_t \circ \phi^{-1}} \circ \phi^t \circ \psi^t,$$

which concludes that $\phi \circ \psi$ is a Hamiltonian diffeomorphism generated by $H_t + G_t \circ \phi^{-1}$. □

(1c-ii) "Physical phase space". Symplectic manifolds should be regarded as generalizations of the notion of phase spaces introduced beforehand, where we don't require the manifold M to be diffeomorphic to the cotangent bundle of some other smooth manifolds. In this paragraph, we will focus on "physical" phase spaces, i.e. cotangent bundle of smooth manifolds, to see the role that symplectic structures play in classical mechanics.

1.12 Example. Before introducing general symplectic structures on physical phase spaces, let's look at the simplest example of symplectic manifolds: the **symplectic space** $(\mathbb{R}^{2n}, \omega_{\text{std}})$ where $\omega_{\text{std}} := \sum_{i=1}^n dq_i \wedge dp_i$ is the standard symplectic 2-form on $\mathbb{R}^{2n}$ with coordinates $(q_1, p_1, \cdots, q_n, p_n)$, regarding (q_i) as the position coordinates and (p_i) as the momentum coordinates. Given any smooth function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$, we can compute that $dH = \displaystyle\sum_{i=1}^n \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i$, and hence the Hamiltonian vector field is given by

$$X_H = \sum_{i=1}^n \left(-\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} + \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} \right).$$

Therefore an integral curve $\gamma(t) = (q_1(t), p_1(t), \cdots, q_n(t), p_n(t))$ of $-X_H$ is a solution of the system of ODEs

$$\begin{cases} \dot{q}_i(t) = \partial_{p_i} H(q(t), p(t)), \\ \dot{p}_i(t) = -\partial_{q_i} H(q(t), p(t)), \end{cases}$$

with given initial condition $\gamma(0) = (q_1(0), p_1(0), \cdots, q_n(0), p_n(0))$. This is exactly the Hamilton's system we have described, and therefore describes the motion of particles in the mechanical system determined by H.

Now let's provide a general construction of symplectic structures on physical phase spaces, T^*M, for any smooth manifold M. We start with a canonical differential 1-form defined on cotangent bundles:

1.13 Definition. The **tautological 1-form** λ_{tau} on T^*M is the differential 1-form in $C^\infty(T^*M, T^*T^*M)$ such that for all differential form $\alpha \in C^\infty(M, T^*M)$, regarded as a section of the cotangent bundle T^*M, we have $\alpha^*\lambda_{\text{tau}} = -\alpha$.

Unfolding the definition, we see for each pair $(q,p) \in T^*M$, we have $(\lambda_{\text{tau}})_{(q,p)} = -pdq$, and therefore $\omega_{\text{std}} := d\lambda_{\text{tau}}$ defines a symplectic structure on T^*M.

1.14 Definition. In T^*M, there is a notable canonical vector field $Z \in C^\infty(T^*M, TT^*M)$, called the **Liouville vector field**, which is the dual vector field of the tautological 1-form λ_{tau}. By Cartan's magical formula we quickly see that $\mathcal{L}_Z\omega_{\text{std}} = \omega_{\text{std}}$, which is not symplectic but exponentially scales the symplectic form ω_{std}.

1.15 Example. Let's revisit the harmonic oscillator. The energy function/hamiltonian for the harmonic oscillator is given by

$$H(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2 \colon T^*\mathbb{R} \to \mathbb{R},$$

and we can therefore write down the system of motions as

$$\begin{cases} \dot{q}(t) = \partial_p H(q(t), p(t)) = \frac{1}{m} p(t), \\ \dot{p}(t) = -\partial_q H(q(t), p(t)) = -kq(t). \end{cases}$$

This is exactly the same system of ODEs we have written down before, and therefore describes the same motion of particles in the harmonic oscillator system.

1.16 Example. Let's also look at the Kepler problem. The energy function/hamiltonian for the Kepler problem is given by

$$H(q,p) = \frac{1}{2m} p^2 - \frac{GM}{|q|} : T^*(\mathbb{R}^3 \setminus \{0\}) \to \mathbb{R},$$

and we can therefore write down the system of motions as

$$\begin{cases} \dot{q}(t) = \partial_p H(q(t), p(t)) = \frac{1}{m} p(t), \\ \dot{p}(t) = -\partial_q H(q(t), p(t)) = -\frac{GM}{|q(t)|^3} q(t). \end{cases}$$

This is exactly the same system of ODEs we have written down before, and therefore describes the same motion of particles in the Kepler problem.

(1c-iii) Integrable systems. From the previous discussions, it's clear that classical mechanical systems can be completely described by a smooth function $H: M \to \mathbb{R}$ on a given symplectic manifold (M, ω), where the symplectic structure ω plays the role of an "intrinsic mechanical structure" that turns a smooth function up to transition (initial energy) into a system of motions where the function describes the energy of the system. Moreover, if we allow initial conditions of the system of motions to vary on M, solving system of motions is equivalent to determining the hamiltonian flow $\phi_H^t : M \to M$ of the hamiltonian vector field X_H of H, and when X_H is complete (true if in particular M is compact), ϕ_H^t is a well-defined family of symplectomorphisms of M for all $t \in \mathbb{R}$.

Solving this system of motions is however still difficult in general, as we need to consider all possible initial conditions, and the behaviour of the integral curve under different initial conditions can be drastically different. In this section we introduce one way to simplify the computations, which probably also appeared in some previous courses that audiences have taken, originally called **first integrals**.

1.17 Definition. Given a system of motions $\dot{\gamma}(t) = X_H(\gamma(t))$ on a symplectic manifold (M, ω), a smooth function $G: M \to \mathbb{R}$ is a **conserved quantity** if $\frac{d}{dt} G(\gamma(t)) = 0$ for all solutions $\gamma(t)$ of the system.

In particular, the Hamiltonian H itself is a conserved quantity. Using our symplectic geometric formulation, we have a more "geometric" description of conserved quantities. Recall that given two vector fields X, Y of a smooth manifold M, the **Lie bracket** $[X, Y]$ of X and Y is the vector field $[X, Y] = XY - YX$, regarded as an action on the space of smooth functions on M.

1.18 Definition. Let $G, K: M \to \mathbb{R}$ be smooth functions on a symplectic manifold (M, ω). We define the **Poisson bracket** of G and K to be the smooth function

$$\{G, K\} \coloneqq \omega(X_G, X_K).$$

1.19 Proposition. $X_{\{G,K\}} = -[X_G, X_K]$.

Proof. Note that

$$d\{G,K\} = d\left(X_K \lrcorner X_G \lrcorner \omega\right) = \mathcal{L}_{X_K}(dG) = \left.\frac{d}{dt}\right|_{t=0} \left(\phi^t_{X_K}\right)^* dG,$$

while on the other hand,

$$([X_G, X_K])\lrcorner\omega = \left(\left.\frac{d}{dt}\right|_{t=0} X_{K\circ\phi^t_{X_G}}\right)\lrcorner\omega = \left.\frac{d}{dt}\right|_{t=0} \left(\phi^t_{X_G}\right)^* dK = -\left.\frac{d}{dt}\right|_{t=0} \left(\phi^t_{X_K}\right)^* dG,$$

and we conclude that $d\{G,K\} = -([X_G, X_K])\lrcorner\omega$. □

1 Exercise. Show that G is a conserved quantity for H if and only if $\{G,H\} = 0$.

It's clear that

2 Exercise. Let G be a conserved quantity for H, then any integral curve of the hamiltonian vector field X_H lies on the level set $G^{-1}(c)$ for some real number c.

Therefore if we can find a conserved quantity G for H, we have a constraint for the system of motions determined by H: it has to lie on one of the real codimension 1 hypersurfaces $\{G(x) = c\}$. Moreover, note that the hamiltonian flow ϕ^t_G of G also preserves the level set $\{G = c\}$, we can take a further quotient of $\{G = c\}$ by identifying trajectories of X_G, which, when c is a regular value of G and ϕ^t_G acts freely, gives us a smooth manifold $M//_cG$ of dimension $2n-2$, which inherits a symplectic structure from M. As the quotient by orbits of X_G preserves values of the hamiltonian H, the function $H\colon M \to \mathbb{R}$ also descends to a smooth function $\bar{H}\colon M//_cG \to \mathbb{R}$, and therefore we have reduced the problem of finding solutions to the system of motions on M to finding solutions to the system of motions on $M//_cG$, which is a symplectic manifold of 2-dimension lower. This process partially solves the equation in some sense, and if we can find n conserved quantities $G_1, \cdots, G_n$ such that $\{G_i, G_j\} = 0$ for all i, j, then we are able to reduce the system of motions to a system of motions on a manifold of dimension 0, and therefore obtain a complete description of the system via these conversed quantities.

1.20 Definition. Under this assumption and further assume that the hamiltonian flows $\phi^t_{G_i}$ of the convserved quantities are defined for all t, then we call the collection $\{G_1, \cdots, G_n\}$ of conserved quantities a **completely integrable system** on M.

Note that each $G_i\colon M \to \mathbb{R}$ provides a natural $\mathbb{R}$-action on M by the hamiltonian flow $\phi^t_{G_i}\colon M \to M$, and as $[X_{G_i}, X_{G_j}] = 0$ for all i, j, the n-tuple $(G_1, \cdots, G_n)$ induces a natural $\mathbb{R}^n$-action on M by the hamiltonian flows $x \mapsto \phi^{t_1}_{G_1} \circ \phi^{t_2}_{G_2} \circ \cdots \circ \phi^{t_n}_{G_n}(x)$ for $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$ and $x \in M$. Moreover, the map $(G_1, \cdots, G_n)\colon M \to \mathbb{R}^n$ is $\mathbb{R}^n$-equivariant, where $\mathbb{R}^n$ acts trivially on the base.

1.21 Theorem (Little Arnold-Liouville Theorem). Let $(G_1, \cdots, G_n)\colon M \to \mathbb{R}^n$ be a completely integrable system and $\sigma\colon \mathbb{R}^n \supseteq U \to M$ a smooth map such that $(G_1, \cdots, G_n)\,\sigma(\vec{c}) = \vec{c}$, then all orbits of $\sigma(\vec{c})$ under the $\mathbb{R}^n$ action are diffeomorphic to $\mathbb{R}^k \times T^{n-k}$ for some $0 \le k \le n$, where $T^{n-k} \cong \mathbb{R}^{n-k}/\Lambda$ is a torus of dimension $n-k$ for some lattice $\mathbb{Z}^{n-k} \cong \Lambda \subseteq \mathbb{R}^{n-k}$.

Proof. It suffices to look at the stabilizer of the $\mathbb{R}^n$-action on a given point $\sigma(\vec{c})$. The existence of σ implies that the differential of the map $(G_1, \cdots, G_n)$ has full rank at $\sigma(\vec{c})$, so that the hamiltonian vector fields $\{X_{G_1}, \cdots, X_{G_n}\}$ have to be linearly independent near $\sigma(\vec{c})$. Therefore the image $U_\varepsilon(0)\sigma(\vec{c})$ of a very small open subset $0 \in U_\varepsilon(0) \subseteq \mathbb{R}^n$ under the $\mathbb{R}^n$-action has to be n-dimensional, hence an open neighbourhood of $\sigma(\vec{c})$ in $(G_1, \cdots, G_n)^{-1}(c)$ as the vector fields have to preserve the fibres. This implies that the stabilizer subgroup $\Lambda_{\sigma(\vec{c})}$ at $\sigma(\vec{c})$ is discrete. Using the time of the hamiltonian flow $\phi_{G_i}^{t_i}$ of G_i as coordinates on $\mathbb{R}^n$, we can identify the stabilizer subgroup $\Lambda_{\sigma(\vec{c})}$ with the **period lattice** $\Lambda_{\sigma(\vec{c})} = \left\{(t_1, \cdots, t_n) \in \mathbb{R}^n \middle| \phi_{G_1}^{t_1} \circ \cdots \circ \phi_{G_n}^{t_n} \sigma(\vec{c}) = \sigma(\vec{c})\right\} \subseteq \mathbb{R}^n$. This concludes the proof. $\square$

1.22 Lemma. Furthermore, if c is a regular value of $\vec{G} = (G_1, \cdots, G_n)$, then all connected components of the fibre $\vec{G}^{-1}(c)$ are orbits of the $\mathbb{R}^n$-action.

Proof. If c is a regular value of $\vec{G}$, then all the points on $\vec{G}^{-1}(c)$ are regular points of $\vec{G}$, i.e. the differential $d\vec{G}$ has full rank. This implies that the $\mathbb{R}^n$-action on any point x of $\vec{G}^{-1}(c)$ provides an open neighbourhood of x in $\vec{G}^{-1}(c)$, so that an $\mathbb{R}^n$-orbit is open in $\vec{G}^{-1}(c)$. On the other hand, it's nor hard to check that an $\mathbb{R}^n$-orbit also has to be closed in $\vec{G}^{-1}(c)$, and therefore it has to coincide with connected components of $\vec{G}^{-1}(c)$. $\square$

1.23 Example. Consider the harmonic oscillator, which under Hamiltonian formalism is given by the smooth function $H(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$ on the symplectic manifold $(T^*\mathbb{R}, \omega_{\text{std}})$. Set for simplicity $m = k = 1$ so that $H(q,p) = \frac{1}{2}(p^2+q^2)$, with corresponding hamiltonian vector field $X_H = -p\frac{\partial}{\partial q} + q\frac{\partial}{\partial p}$. H is clearly a conserved quantity of this system, and since $\dim_{\mathbb{R}} T^*\mathbb{R} = 2$, $H\colon T^*\mathbb{R} \to \mathbb{R}$ forms a completely integrable system. Note that $(0,0)$ is a critical point of H, so that the regular values of H lie on the open interval $(0, +\infty) \subseteq \mathbb{R}$, where fibres are of the form

$$H^{-1}(c) = \left\{(q,p) \in T^*\mathbb{R} \middle| q^2 + p^2 = 2c\right\} \cong \mathbb{S}^1.$$

The integral curve of X_H is computed in Example 1.4: they are of the form $\gamma(t) = (a\cos t + b\sin t, -a\sin t + b\cos t)$ for some chosen constant $a, b \in \mathbb{R}$. We assume further that $\gamma(0) = (\sqrt{2c}, 0)$ so that $a = \sqrt{2c}$ and $b = 0$. It's then clear that $\{\gamma(t)\}_{t\in\mathbb{R}}$ covers the whole fibre $H^{-1}(c)$, with relation $\gamma(2\pi) = \gamma(0)$, and hence the period lattice at $(\sqrt{2c}, 0)$ is $2\pi\mathbb{Z} \subseteq \mathbb{R}$. It's not hard to check that the period lattices at any point of $H^{-1}(c)$, $c > 0$, are all given by $2\pi\mathbb{Z}$.

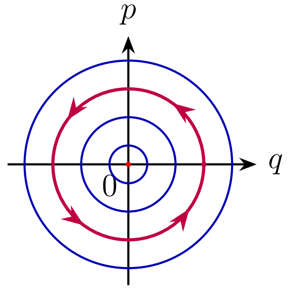

Figure 1: The phase space of the harmonic oscillator, where fibres of H are circles and the flow of X_H is periodic with period 2π.

As we can assign to each $x \in M$ a period lattice $\Lambda_x^{\vec{G}}$ in $\mathbb{R}^n$, patching all the lattices at each point of M provides us a subspace $\Lambda^{\vec{G}} \subseteq \mathbb{R}^n \times M$. This subspace is often what we refer to as the **period lattice** of the completely integrable system $\vec{G}\colon M \to \mathbb{R}^n$.

1.24 Definition. The period lattice $\Lambda^{\vec{G}}$ is **standard** if the projection $\Lambda^{\vec{G}} \to \mathbb{R}^n$ is the constant lattice $2\pi\mathbb{Z}^{n-k} \subseteq \mathbb{R}^n$.

Assume that fibres of $\vec{G}\colon M \to \mathbb{R}^n$ are all regular and connected, then Lemma 1.22 implies that the $\mathbb{R}^n$-action on M is transitive on each fibre, and hence the period lattice $\Lambda^{\vec{G}}$ descends to a subspace $\overline{\Lambda}^{\vec{G}} \subseteq \mathbb{R}^n \times B$ over $B \coloneqq \vec{G}(M) \subseteq \mathbb{R}^n$ whose fibre at each point $\vec{b} \in B$ is a lattice $\overline{\Lambda}_{\vec{b}}^{\vec{G}} \subseteq \mathbb{R}^n$.

2. Flexibility: Moser's Argument

The first classes of properties we saw about symplectic manifolds in history are closer to smooth topology, which we now call **flexibility** properties. Such properties were originally discovered by Moser [Mos65] for volume-preserving diffeomorphisms, whose arguments were easily adapted to symplectic manifolds. Several results concerning flexibility properties of symplectic maifolds were then quickly discovered, for instance neighbourhood theorems for symplectic and several special types of submanifolds of a given symplectic manifold, where we can arrange so that the symplectic structure in this neighbourhood is standard. As an outcome, there was a famous slogan which was probably known to many of the readers: **"Symplectic geometry has no local invariants"**, comparing to Riemannian manifolds where sectional curvatures obstruct us from establishing neighbourhood theorems.

In this section we will go through Moser's technique and prove several neighbourhood theorems for symplectic manifolds. Before going into the details, let's first have a look at some natural "non-physical" phase spaces, which arises from the study of complex manifolds.

(2a) Complex and almost-Kähler manifolds. Holomorphic geometry is not of major concern in this course, but the shadow of complex geometry will keep arising during the lecture, so let's spend a little bit of time to introduce some of the basic definitions. Readers who want to know more about complex geometry can go to D. Huybrechts [Huy05] for help.

2.1 Definition. A **complex manifold** is defined very similarly to a smooth manifold, where we look at a topological space M so that it's locally diffeomorphic to $\mathbb{C}^n$ for some uniform n, and the transition function between different local charts are required to have complex linear differentials. Equivalently, they should all be holomorphic.

There is a special class of complex manifolds called **Kähler manifolds**, which are complex manifolds M equipped with a "good metric", i.e. a Riemannian metric g that is compatible with the complex structure J on M in the sense that $g(JX, JY) = g(X, Y)$ for all vector fields X, Y on M. The Riemannian metric g together with the complex structure J provides us a symplectic structure ω on M by the formula $\omega(X, Y) = g(JX, Y)$, which is usually referred to the **Kähler form** of the Kähler manifold (M, g, J).

2.2 Example. The trivial example of the complex vector space $\mathbb{C}^n$ with the standard symplectic structure $\omega_{\text{std}} = \sum_{i=1}^n dx_i \wedge dy_i = \frac{\sqrt{-1}}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$ is clearly a Kähler manifold.

Let's look at a non-trivial example. Consider the quotient space

$$\mathbb{C}P^n := \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \mathbb{C}^\times,$$

where the group $\mathbb{C}^\times$ acts on $\mathbb{C}^{n+1} \setminus \{0\}$ by diagonal scalar multiplication. Recalling the proof that $\mathbb{R}P^n$ is a smooth manifold shows that $\mathbb{C}P^n$ is a complex manifold of dimension n, with canonical open cover

$$U_i = \{[z_0 : z_1 : \cdots : z_n] | z_i = 1\} \cong \mathbb{C}^n.$$

The symplectic structure on $\mathbb{C}P^n$ whose restriction to each U_i is given by

$$\omega_{\text{FS}}|_{U_i} := \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log\left(1 + \sum_{j \neq i} |z_j|^2\right)$$

is a Kähler form on $\mathbb{C}P^n$, called the **Fubini-Study form** on $\mathbb{C}P^n$.

Let's look at the special case when $n = 1$. In this case we only have two charts $U_1 \cong \mathbb{C}_w$ and $U_0 \cong \mathbb{C}_z$ with

$$\omega_{FS}|_{U_0} = \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\left(1+|w|^2\right) = \frac{\sqrt{-1}}{2}\partial\frac{wd\bar{w}}{1+|w|^2} = \frac{\sqrt{-1}}{2}\frac{dw\wedge d\bar{w}}{(1+|w|^2)^2},$$

and similarly

$$\omega_{FS}|_{U_1} = \frac{\sqrt{-1}}{2}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2}.$$

Inside $U_0 \cap U_1 = \{[w:z]|w \neq 0, z \neq 0\} \cong \mathbb{C}^*$ where w is identified with z^{-1} under this identification, we have $\omega_{FS}|_{U_0}|_{U_0\cap U_1} = \frac{\sqrt{-1}}{2}\frac{dz\wedge d\bar{z}}{(1+|z|^{-2})^2}\frac{1}{|z|^4} = \frac{\sqrt{-1}}{2}\frac{dz\wedge d\bar{z}}{(1+|z|^2)^2} = \omega_{FS}|_{U_1}|_{U_0\cap U_1}$, so that ω_{FS} is well-defined on the whole $\mathbb{C}P^1 \cong \mathbb{S}^2$.

To compute the volume of $\mathbb{P}^1$, it suffices to restrict our attention to $U_0 \subseteq \mathbb{P}^1$ as the complement clearly has measure 0. We can then integrate directly to get

$$\int_{\mathbb{C}}\frac{\sqrt{-1}}{2}\frac{1}{(1+|z|^2)^2}dz\wedge d\bar{z} = \int_0^{2\pi}\int_0^{\infty}\frac{r}{(1+r^2)^2}drd\theta = \pi.$$

2.3 Example. Let $M \hookrightarrow \mathbb{C}P^n$ be a smooth complex manifold holomorphically embedded into $\mathbb{C}P^n$, then the restriction of ω_{FS} to M induces a Kähler form on M, which we still call the **Fubini-Study form** on M.

There is then a natural question of comparing between symplectic and Kähler manifolds, and the answer is clear: there're much more symplectic manifolds than Kähler ones, which we will repeatly learn during the lecture. However, any symplectic manifold is automatically "almost Kähler", which we will discuss a little bit now.

2.4 Definition. An **almost complex structure** on a smooth manifold M is a smooth vector bundle endomorphism $J\colon TM \to TM$ such that $J^2 = -\mathrm{Id}$. A symplectic structure ω on M is said to be **compatible** with the almost complex structure J if $\omega(JX, JY) = \omega(X,Y)$ for all vector fields X, Y on M, and $g(X,Y) := \omega(JX,Y)$ defines a Riemannian metric on M. In this case, we call the triple (M, ω, J) an **almost Kähler manifold**.

2.5 Proposition. A symplectic manifold (M, ω) is always almost Kähler. In other words, there exists an almost complex structure J on M which is compatible with ω. Moreover, the space of such almost complex structures is contractible.

Proof. The proof roughly follows [Gro85]. For more details, see [AL94]. We first handle the local case when $M = \mathbb{R}^{2n}$ and $\omega = \omega_{\mathrm{std}}$ is the standard symplectic structure on $\mathbb{R}^{2n}$. In this case, a linear almost complex structure is a matrix $J \in \mathrm{End}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ satisfying $J^2 = -\mathrm{Id}$ and $J^T\omega_{\mathrm{std}}J = \omega_{\mathrm{std}}$. Write $\mathcal{J}(\mathbb{R}^{2n})$ to be the space of all complex structures on $\mathbb{R}^{2n}$, and let $J_0 = \omega_{\mathrm{std}}$ be the **standard complex structure**, then we have

2.6 Lemma. The map $J \mapsto (J + J_0)^{-1} \circ (J - J_0)$ is a diffeomorphism from $\mathcal{J}(\mathbb{R}^{2n})$ onto the open unit ball in the vector space of symmetric matrices S such that $J_0S + SJ_0 = 0$. Here the topology is given by the operator norm

$$\|S\| = \inf\{c|\omega(Sx, J_0Sx) \leq c\omega(x, J_0x)\ \forall x\}.$$

This readily implies that the space of linear complex structures on $\mathbb{R}^{2n}$ is contractible.

For general manifold M, the tangent bundle $TM \to M$ is a symplectic vector bundle, meaning the symplectic structure ω_M provides a symplectic structure on each fibre T_pM for all $p \in M$. Therefore the space of complex structures $\mathcal{J}(T_pM)$ on the vector space T_pM is contractible. By taking the space of linear complex structures on each tangent space we get a fibre bundle $\mathcal{J}(M) \to M$ over M, whose fibres are contractible by Lemma 2.6.

Fibre bundles over a smooth manifold M with contractible fibres always admit global sections, and the space of sections are contractible. To show this, first note that M admits a CW decomposition, so that we can extend sections inductively from 0-skeleton to the top skeleton, by choosing arbitrary almost complex structures on each vertex, and for edges connecting vertices, choosing any path connecting the corresponding almost complex structures chosen on the vertex. We then see that obstructions to extending sections from k-skeleton to $(k+1)$-skeleton come from homotopy groups of $\mathcal{J}(\mathbb{R}^{2n})$, which are trivial since it's contractible. Therefore we always obtain a global continuous section of $\mathcal{J}(M)$, which can be perturbed to be smooth due to approximation theorems [Hir76, Theorem 2.2]. Isotopies between sections can be regarded as sections over the product manifold $M \times [0,1]$ with fixed boundary conditions, and the same argument applies. For more details on obstruction theory, see [Whi78]. $\square$

Let's finish the proof of the lemma.

Proof of Lemma 2.6. This is proved by a direct verification. As J and J_0 are all complex structures, we have $\omega(x,(J+J_0)x) > 0$ if $x \neq 0$, so $J + J_0$ is invertible. Since $g(x,y) := \omega(x, Jy)$ defines a symmetric bilinear form, we must have $(\omega_{\mathrm{std}} J)^T = \omega_{\mathrm{std}} J$, while $\omega_{\mathrm{std}} = J_0$ satisfies $J_0^{-1} = -J_0$. Therefore $(J+J_0)^{-1} \circ (J - J_0) = (J_0^{-1}J + \mathrm{Id})^{-1} \circ (J_0^{-1}J - \mathrm{Id})$ is symmetric. Write $S = (J+J_0)^{-1} \circ (J-J_0) = (J_0^{-1}J+\mathrm{Id})^{-1}(J_0^{-1}J-\mathrm{Id}) = (A+\mathrm{Id})^{-1}(A-\mathrm{Id})$, then the fact that $\|S\| < 1$ is equivalent to $\|Ax - x\|^2 < \|Ax + x\|^2$ for all $x \neq 0$. Let's check that

$$\|Ax + x\|^2 - \|Ax - x\|^2 = 4\omega(x, J_0 Ax) = 4\omega(x, Jx) > 0.$$

Therefore the image should satisfy $\|S\| < 1$. To verify the last identity, note that

$$\omega(Sx,y) + \omega(x,Sy) = \frac{1}{2}\left(\omega((\mathrm{Id}+S)x, (\mathrm{Id}+S)y) - \omega((\mathrm{Id}-S)x, (\mathrm{Id}-S)y)\right),$$

where $\omega((\mathrm{Id}+S)x, (\mathrm{Id}+S)y) = \omega(2Ax, 2Ay) = 4\omega(x,y)$ and $\omega((\mathrm{Id}-S)x, (\mathrm{Id}-S)y) = \omega(2x, 2y) = 4\omega(x,y)$. Since this holds for all (x,y), by non-degeneracy of ω we have $SJ_0 + J_0 S = 0$.

Finally, for each symmetric S with given conditions we know that $\mathrm{Id} - S$ is invertible so that

$$J = J_0(\mathrm{Id}+S) \circ (\mathrm{Id}-S)^{-1}$$

defines an endomorphism of $\mathbb{R}^{2n}$. We can check that $J^2 = -\mathrm{Id}$, $J^T \omega_{\mathrm{std}} J = \omega_{\mathrm{std}}$ and $(J\omega)^T = J\omega$ by direct computations, which we omit here. $\square$

However, the converse is not true. These are far away from what we want to discuss in this course, so we will only give a brief overview. The existence of almost complex structures on a general smooth manifold of even dimension has topological obstructions, coming from the fact that with such an almost complex structure the tangent bundle

TM is a complex vector bundle, so that its **first Chern class** $c_1(TM) \in H^2(M;\mathbb{Z})$ is well-defined. When $\dim M = 4$, there is an identity

$$\langle c_1^2(TM), [M]\rangle = 2\chi(M) + 3\sigma(M),$$

where $\chi(M)$ is the Euler characteristic of M and $\sigma(M)$ is the signature of M. It was further proved that

2.7 THEOREM (Wu [Wu52]). Homotopy classes of almost complex structures on a smooth manifold M of dimension 4 are in one-to-one correspondence with the elements $c \in H^2(M;\mathbb{Z})$ lifting $w_2(TM) \in H^2(M;\mathbb{Z}/2)$ and such that $\langle c^2, [M]\rangle = 2\chi(M) + 3\sigma(M)$.

In the same paper of Wu [Wu52], he also constructed an almost complex structure on the 6-sphere $\mathbb{S}^6$, and showed that the only even-dimensional spheres that can have almost complex structures are $\mathbb{S}^2$ and $\mathbb{S}^6$.

3 EXERCISE. Show that $\mathbb{S}^6$ is not a symplectic manifold.

Hint: a symplectic structure is a priori a closed 2-form. Look at the de Rham cohomology of $\mathbb{S}^6$.

Therefore $\mathbb{S}^6$ is an example of an almost complex manifold that is not almost Kähler.

There're also obstructions from upgrading any almost Kähler manifold to a Kähler manifold, which is provided by the **Nijenhuis tensor** [NN57, Che55]

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

for vector fields $X, Y \in C^\infty(M, TM)$, which can be regarded also as failures of J being parallel with respect to the Levi-Civita connection of the metric $g(X, Y) = \omega(JX, Y)$. The theorem of Newlander-Nirenberg [NN57] showed that whenever the Nijenhuis tensor vanishes, we can find an integrable complex structure on M making M into a Kähler manifold.

(2b) Moser's isotopy. Now we step into flexibility theorems for symplectic manifolds. The proof patterns are similar so we will only go through a few proofs and the remainings are left as exercises.

2.8 PROPOSITION. Let M be a smooth even-dimensional manifold and $Q \subseteq M$ a compact submanifold. Assume that ω_0 and ω_1 are two symplectic forms on M such that $\omega_0|_Q = \omega_1|_Q$ and $[\omega_0] = [\omega_1] \in H^2(M;\mathbb{R})$, then there exists open neighbourhoods $\mathrm{Nbd}_0(Q)$ and $\mathrm{Nbd}_1(Q)$ of Q in M and a diffeomorphism $\psi\colon \mathrm{Nbd}_0(Q) \to \mathrm{Nbd}_1(Q)$ such that $\psi^*\omega_1 = \omega_0$ and $\psi|_Q = \mathrm{Id}_Q$.

Proof. As Q is a closed smooth submanifold of M, the restriction $i^*TM \to Q$ of the tangent bundle of M to Q under the inclusion $i\colon Q \hookrightarrow M$ contains TQ as a subbundle, with quotient $N_MQ = i^*TM/TQ$. Pick a Riemannian metric on M so that N_MQ can be embedded back into i^*TM as the orthogonal complement of TQ.

Under the Riemannian metric, for each $q \in Q$, the exponential map [Cha93, Theorem I.3.2] $\exp_q\colon T_qM \to M$ determines a diffeomorphism from an open neighbourhood of q in T_qM to an open neighbourhood of q in M, whose restriction to T_qQ has image in Q. With local coordinate charts of M subordinate to the embedding $Q \hookrightarrow M$, we can see that $\exp^\perp\colon N_MQ \to M$ sending $(N_MQ)_q$ via $\exp_q$ to M is smooth. Since $\exp_q$

is a diffeomorphism onto its image in an open neighbourhood of q, we can also check that $\exp^{\perp}$ is a diffeomorphism from an open neighbourhood of $Q \hookrightarrow N_M Q$ to an open neighbourhood of $Q \hookrightarrow M$.

We can therefore pull back the symplectic structure ω_0 and ω_1 to an open neighbourhood of Q in $N_M Q$, which by abuse of notations still denoted by ω_0 and ω_1. Consider the family $\omega_t = t\omega_1 + (1-t)\omega_0$, then ω_t is a family of symplectic forms in an open neighbourhood of Q in $N_M Q$ with $[\omega_t] = [\omega_0] = [\omega_1]$ for all $t \in [0,1]$.

We look for family of diffeomorphisms $\{\psi_t\colon \mathrm{Nbd}_{N_M Q}(Q) \to N_M Q\}_{0\le t\le 1}$ such that $\psi_0 = \mathrm{Id}$, $\psi_t|_Q = \mathrm{Id}$ and $\psi_t^*\omega_t = \omega_0$ for all $t \in [0,1]$. Differentiating the equation $\psi_t^*\omega_t = \omega_0$ with respect to t, we get

$$\psi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) = 0,$$

where X_t is the derivative of the diffeomorphism ψ_t with respect to t, or in other words the vector field $d\psi_t(\partial/\partial t)$ regarding $\psi_t\colon \mathrm{Nbd}_{N_M Q}(Q) \times [0,1]_t \to N_M Q$ as a map on the product. Therefore we must have

$$\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t = 0.$$

By Cartan's magical formula, $\mathcal{L}_{X_t}\omega_t = d(X_t \lrcorner \omega_t) + X_t \lrcorner d\omega_t = d(X_t \lrcorner \omega_t) \coloneqq d\sigma_t$ and $\dfrac{d}{dt}\omega_t = \omega_1 - \omega_0$. As $[\omega_1] = [\omega_0]$, the derivative $\dot{\omega}_t = \omega_1 - \omega_0$ is exact and hence this equation has a solution σ_t.

However, it's a problem whether this σ_t can be chosen to satisfy $\sigma_t|_Q = 0$, and we provide here a direct construction via **integration over fibres**. Consider a family of maps $\psi^t\colon N_M Q \times [0,1] \to M$ given by

$$\psi^t(q,v) = \exp_q^{\perp}(q, tv),$$

then for a chosen neighbourhood $\mathrm{Nbd}_{N_M Q}(Q)$, ψ^t is a diffeomorphism onto its image whenever $t \neq 0$, and ψ^0 contracts $\mathrm{Nbd}_{N_M Q}(Q)$ to Q. Up to identifying a neighbourhood of Q with an open subset of M, we also get $\psi^1 = \mathrm{Id}$. Therefore we can compute that $(\psi^1)^*(\dot{\omega}_t) = \dot{\omega}_t$ and $(\psi^0)^*\dot{\omega}_t = 0$. Cartan's magical formula implies that

$$\frac{d}{dt}(\psi^t)^*\dot{\omega}_t = (\psi^t)^*\left(\mathcal{L}_{X_t}\dot{\omega}_t\right) = d\left((\psi^t)^*(X_t \lrcorner \dot{\omega}_t)\right) = d\sigma_t,$$

where $\sigma_t \coloneqq (\psi^t)^*(X_t \lrcorner \dot{\omega}_t)$ is a family of 1-forms determined by the vector field $X_t = \left(\dfrac{d}{dt}\psi^t\right) \circ \psi^{-t}$ is smooth in t. Though X_t is not defined at $t = 0$, but we can check directly that σ_t is still smooth for $t \in [0,1]$[1]. Now by Newton-Leibnitz formula,

$$\dot{\omega}_t = (\psi^1)^*\dot{\omega}_t - (\psi^0)^*\dot{\omega}_t = \int_0^1 \frac{d}{dt}(\psi^t)^*\dot{\omega}_t dt = \int_0^1 d\sigma_t dt = d\left(\int_0^1 \sigma_t dt\right) = d\sigma,$$

with $\sigma|_Q = 0$. Since ω_t is non-degenerate, σ induces a unique smooth vector field X_t so that $\sigma = X_t \lrcorner \omega_t$. As $\sigma|_Q = 0$, we have $X_t|_Q = 0$, and hence the flow $\phi_{X_t}^t$ of X_t restricts to identity on Q for all $t \in [0,1]$, with $\phi_{X_t}^0 = \mathrm{Id}$ and $(\phi_{X_t}^1)^*\omega_1 = \omega_0$ as discussed.

Note that the flow $\phi_{X_t}^1$ is not defined everywhere in $\mathrm{Nbd}_{N_M Q}(Q)$, but as $\phi_{X_t}^t(Q) = Q$, we can choose a smaller neighbourhood $\mathrm{Nbd}_0(Q)$ of Q in $N_M Q$ such that $\phi_{X_t}^1$ is well-defined, and write the image of $\phi_{X_t}^1$ as $\mathrm{Nbd}_1(Q)$. This is the desired diffeomorphism. $\square$

[1] We need a computation of differential of exponential maps along radial vectors, and the result is simple. See any Riemannian geometry books, e.g. [Cha93].

Set Q to be a point, then we get the famous **Darboux neighbourhood theorem**

2.9 THEOREM (Darboux). Let (M, ω) be a symplectic manifold and $p \in M$ any point, then there exists an open neighbourhood $p \in U \subseteq M$ of p in M, an open neighbourhood $0 \in U \subseteq \mathbb{R}^{2n}$ of the origin in $\mathbb{R}^{2n}$ and a diffeomorphism $\psi\colon U \to V$ such that $\psi^*\omega = \omega_{\text{std}}$ and $\psi(p) = 0$.

4 EXERCISE. Prove Darboux' theorem directly without referring to Proposition 2.8.

This procedure of finding diffeomorphisms from constructing dual vector fields originated from Moser's work [Mos65], and this approach can be found in the great book of McDuff-Salamon [MS17]. Another application of Moser's argument shows that symplectic structures on compact symplectic manifolds are completely determined by their cohomology classes $H^2_{dR}(X; \mathbb{R})$.

2.10 THEOREM. Let M be a compact smooth manifold of dimension $2n$ and $\{\omega_t\}_{0\leq t\leq 1}$ be a smooth family of cohomologous symplectic forms on M, then there exists a family of diffeomorphisms $\{\psi^t\colon M \to M\}_{0\leq t\leq 1}$ such that $(\psi^t)^*\omega_t = \omega_0$ for all $t \in [0, 1]$.

Proof. We omit the procedure of running Moser's argument, and it suffices for us to check that there exists a smooth family of 1-forms σ_t such that $\dot{\omega}_t = d\sigma_t$. To achieve this, we consider an open cover of M where any of the finite intersections can be identified with an open subset of $\mathbb{R}^{2n}$ [BT82, Theorem 5.1], and pick a partition of unity subordinate to this open cover so that we can assume the two-forms $\dot{\omega}_t$ are supported in a compact subset in $\mathbb{R}^{2n}$. We can then pick a point p inside the support and run the integration over fibres as in Proposition 2.8 to get a smooth family of 1-forms σ_t such that $\dot{\omega}_t = d\sigma_t$.[2] For two such charts U, V we can find two a priori different families of 1-forms σ_t^U and σ_t^V, but they differ in $U \cap V$ by a family of exact 1-forms, so we can run the integration over fibres again to get a correction family of exact 1-forms σ_t^{UV} such that $\sigma_t^U - \sigma_t^V = \sigma_t^{UV}$, and therefore by adding this 1-form to one of the two families we get a family of 1-forms σ_t defined on the union $U \cup V$. We can therefore get a global 1-form σ_t defined on M by induction. □

(2c) Special submanifolds. After introducing the basics of Moser's argument, let's look at its effectivity by proving several neighbourhood theorems for certain special type of submanifolds of a given symplectic manifold (M, ω).

2.11 DEFINITION. A smooth submanifold $N \subseteq M$ of a symplectic manifold (M, ω) is **symplectic** if the restriction $\omega|_N$ is a symplectic structure on N.

A vector bundle $E \to N$ is a **symplectic vector bundle** if there exists a smooth family of non-degenerate skew-symmetric bilinear forms $\omega_p\colon E_p \times E_p \to \mathbb{R}$ for all $p \in N$.

In particular, a symplectic structure on M induces a symplectic structure on the tangent bundle TM, so the restriction $TM|_N$ of the tangent bundle of M to N is a symplectic vector bundle over N, whose subbundle $TN \hookrightarrow TM|_N$ is also symplectic.

2.12 DEFINITION. The symplectic orthogonal complement TN^ω of TN in $TM|_N$ is called the **symplectic conormal bundle** of N in M. It's clear that $(TN)^\omega$ is also a symplectic vector bundle over N.

[2]Note that the 1-forms σ_t are timewise provided by integrations with respect to the exponential maps, and the smooth dependence on t can be checked directly from the formula.

Writte $\pi\colon TN^\omega \to N$ be the bundle map, and write $\omega_N = \omega|_N$ and $\Omega = \omega|_{TN^\omega}$ be the restricted symplectic structures from M and TM to N and TN^ω respectively, then we can check that $\pi^*\omega_N + \Omega$ defines a symplectic structure on the total space of the symplectic conormal bundle TN^ω, which we write as ω_{TN^ω}.

5 Exercise. Prove the following neighbourhood theorem: for $N \subseteq M$ a compact symplectic submanifold, there exists an open neighbourhood $\mathrm{Nbd}_M(N)$ of N in M and an open neighbourhood $\mathrm{Nbd}_{TN^\omega}(N)$ of the zero section $N \hookrightarrow TN^\omega$ in the symplectic conormal bundle TN^ω and a diffeomorphism $\psi\colon \mathrm{Nbd}_{TN^\omega}(N) \to \mathrm{Nbd}_M(N)$ such that $\psi|_N = \mathrm{Id}$ and $\psi^*\omega = \omega_{TN^\omega}$.

Despite symplectic submanifolds, there're some other special submanifolds that we are interested in.

2.13 Definition. A smooth submanifold $N \subseteq M$ of a symplectic manifold (M, ω) is **isotropic** if the restriction $\omega|_N$ is identically zero, or equivalently $TN \subseteq TN^\omega$.

A smooth submanifold $N \subseteq M$ of a symplectic manifold (M, ω) is **coisotropic** if $TN^\omega \subseteq TN$.

A smooth submanifold $N \subseteq M$ of a symplectic manifold (M, ω) is **Lagrangian** if $TN = TN^\omega$, or equivalently N is both isotropic and coisotropic.

6 Exercise. Show that an isotropic submanifold $N \subseteq M^{2n}$ has to have dimension at most n, and a coisotropic submanifold $N \subseteq M^{2n}$ has to have dimension at least n. In particular, a Lagrangian submanifold $N \subseteq M^{2n}$ has to be half-dimensional.

7 Exercise. Show that a real hypersurface $C \subseteq M$ of a symplectic manifold M is always coisotropic. Similarly, a one-dimensional submanifold $I \subseteq M$ is always isotropic.

2.14 Example. For any smooth manifold L, the "physical phase space" T^*L has natural Lagrangian submanifolds: the zero section $L \hookrightarrow T^*L$ and any cotangent fibres $T^*_qL \hookrightarrow T^*L$ for $q \in L$.

8 Exercise. Inside the cotangent bundle T^*L, find the equivalence condition for a graph

$$\Gamma(\alpha) := \{(x, \alpha_x) \in T^*L | x \in L\}$$

of a differential 1-form $\alpha \in C^\infty(L, T^*L)$ to be Lagrangian in T^*L.

2.15 Definition. Two Lagrangian submanifolds $L_0, L_1 \subseteq M$ are **Lagrangian isotopic** if there is a smooth family of Lagrangian submanifolds $\{L_t\}_{0\le t\le 1}$ in M. They are furthermore **hamiltonian isotopic** if there exists a hamiltonian diffeomorphism $\phi \in \mathrm{Ham}(M, \omega)$ such that $\phi(L_0) = L_1$.

9 Exercise. Show that $\Gamma(\alpha)$, once being Lagrangian, is always Lagrangian isotopic to the zero section. Show that it is furthermore hamiltonian isotopic to the zero

section $L \hookrightarrow T^*L$ if and only if α is an exact 1-form.

10 Exercise. Show that for any compact Lagrangian submanifold $L \hookrightarrow M$ of a symplectic manifold (M, ω), there exists an open neighbourhood $\mathrm{Nbd}_M(L)$ of L in M and an open neighbourhood $\mathrm{Nbd}_{T^*L}(L)$ of the zero section $L \hookrightarrow T^*L$ in the cotangent bundle T^*L and a diffeomorphism $\psi\colon \mathrm{Nbd}_{T^*L}(L) \to \mathrm{Nbd}_M(L)$ such that $\psi|_L = \mathrm{Id}$ and $\psi^*\omega = \omega_{\mathrm{std}}$.

Hint. The difficult part is to identify the normal bundle of L to M with T^*L and find an embedding $T^*L \supseteq U \hookrightarrow M$ such that the pullback of ω restricts to ω_{std} along the zero section $L \hookrightarrow T^*L$. Pick an almost complex structure J on M compatible with ω, then we can check that $JTL \subseteq TM|_L$ is a Lagrangian subbundle of $TM|_L$ that is transversal to TL, and since J is an isomorphism of bundles, we see that $JTL \cong TL$. Let g_J be the Riemannian metric defined by ω and J, then g_J provides the identification of JTL with T^*L, the cotangent bundle of L. We write $\Phi\colon T^*L \to TL$ be the corresponding isomorphism, and via exponential map we define the map $\phi\colon T^*L \to M$ by

$$\phi(q, \alpha) = \exp_q(J_q\Phi_q(\alpha)).$$

By definition, for $v = v_0 + v_1$ under the identification $TT^*L \cong TL \oplus T^*L$ we have $d\phi_{(q,\alpha)}(v) = v_0 + J_q\Phi_q(v_1)$, and hence

$$\begin{aligned}\phi^*\omega_{(q,0)}(v, w) &= \omega_q(v_0 + J_q\Phi_q(v_1), w_0 + J_q\Phi_q(w_1)) = \omega_q(v_0, J_q\Phi_q(w_1)) - \omega_q(w_0, J_q\Phi_q(v_1)) \\ &= w_1(v_0) - w_0(v_1) = \omega_{\mathrm{std}}(v, w).\end{aligned}$$

This implies that $\phi^*\omega$ and ω_{std} are identified along L, and hence we can apply Moser's argument to finish the proof. □

For isotropic/coisotropic submanifolds, we need to provide a good description of the local model. Let $L \subseteq M$ be an isotropic submanifold, then $TL \subseteq TM$ has symplectic orthogonal complement

$$TL \subseteq (TL)^\omega,$$

whose quotient $(TL)^\omega/TL$ is a symplectic vector bundle over L with the inherited symplectic structure $\overline{\omega}$. Pick an almost complex structure J on TM compatible with ω and let g_J be the corresponding Riemannian metric, we then have direct sum decomposition

$$TM|_L = TL \oplus JTL \oplus E,$$

where $E = (TL \oplus JTL)^\omega$ is the symplectic vector bundle isomorphic to $(TL)^\omega/TL$ and JTL is an isotropic subbundle isomorphic to $TM/(TL)^\omega$. Under this identification, write $\omega_E = \omega|_E$ and identify JTL with T^*L via the metric g_J, we can construct a symplectic structure ω_J on $TM|_L/TL = T^*L \oplus E$ as follows: for any pair of tangent vectors $v = v_0 + v_1 + v_2$ and $w = w_0 + w_1 + w_2$ where $v_0, w_0 \in TL$, $v_1, w_1 \in T^*L$ and $v_2, w_2 \in E$, we define

$$\omega_J(v, w) = v_1(w_0) - w_0(v_1) + \omega_E(v_2, w_2).$$

Then we can check that ω_J is a symplectic structure on $TM|_L/TL$.

11 Exercise. Show that for any compact isotropic submanifold $L \hookrightarrow M$ of a symplectic manifold (M, ω), there exists an open neighbourhood $\mathrm{Nbd}_M(L)$ of L in M and an open neighbourhood $\mathrm{Nbd}_{TM|_L/TL}(L)$ of the zero section $L \hookrightarrow TM|_L/TL$ in the quotient bundle $TM|_L/TL$ and a diffeomorphism $\psi\colon \mathrm{Nbd}_{TM|_L/TL}(L) \to \mathrm{Nbd}_M(L)$ such that $\psi|_L = \mathrm{Id}$ and $\psi^*\omega = \omega_J$ for some chosen almost complex structure J on M.

12 Exercise. Set up the correct local model for a coisotropic submanifold $C \subseteq M$, state and proof the corresponding neighbourhood theorem for coisotropic submanifolds.

Hint: For those who want to directly see the whole procedure, check [Got82].

(2d) Arnold-Liouville theorems continued. The introduction of special submanifolds allows us to provide a better description of completely integrable systems. First of all, we have

2.16 Proposition. Let (M, ω) be a symplectic manifold with a Poisson commute collection of smooth functions $\{H_1, \cdots, H_k\}$. Assume that hamiltonian flows $\phi^t_{H_i}$ of H_i are defined for all $t \in \mathbb{R}$, giving a $\mathbb{R}^k$-action on M, then the orbits of the $\mathbb{R}^k$-action are isotropic and the level sets $\vec{H}^{-1}(c)$ for $c \in \mathbb{R}^k$ regular values are coisotropic.

Proof. The first claim follows from the commutativity and the second claim follows from from the fact that $T\vec{H}^{-1}(c)$ is the zero set in TM of the differential forms dH_i for $i = 1, \cdots, k$, which is the intersection of the symplectic orthogonal complements $\mathrm{span}\{X^{\omega}_{H_i}\}_{i=1}^k$ of the corresponding Hamiltonian vector fields. □

2.17 Corollary. In particular, for $\vec{H}\colon M \to \mathbb{R}^n$ a completely integrable system with regular fibres, the orbits of the $\mathbb{R}^n$-action are Lagrangian submanifolds of M.

Upon determining all the orbits of the completely integrable system, we want to understand how these orbits are patched together to form the symplectic manifold M. Locally around regular orbits, we have the following **Liouville coordinates**:

2.18 Proposition. Let $\vec{H}\colon M \to \mathbb{R}^n$ be a completely integrable system and let $b \in \mathbb{R}^n$ be a regular value of $\vec{H}$. Let $b \in U$ be an open neighbourhood of b in $\mathbb{R}^n$ and $\sigma\colon U \to M$ a smooth **Lagrangian section** of $\vec{H}$, i.e. $\vec{H} \circ \sigma = \mathrm{Id}_U$ and $\sigma(U)$ is a Lagrangian submanifold of M. Then the map

$$\Psi\colon U \times \mathbb{R}^n \to X, \quad \Psi(b, t) = \phi^{t_1}_{H_1} \circ \cdots \circ \phi^{t_n}_{H_n}(\sigma(b))$$

is both an immersion and a submersion such that $\Psi^*\omega = \sum_{i=1}^n db_i \wedge dt_i$, where $(b_1, \cdots, b_n)$ is the standard coordinate in $U \subseteq \mathbb{R}^n$.

Proof. This is basically a computation: write ∂_{b_i} and ∂_{t_i} to be the corresponding vector fields on $U \times \mathbb{R}^n$, then for $1 \leq i \leq n$ it's clear that $\Psi_*(\partial_{b_i})$ has image in $T\sigma(U)$, which is a Lagrangian submanifold and hence $\Psi^*\omega(\partial_{b_i}, \partial_{b_j}) = 0$ for all $1 \leq i, j \leq n$. As all $\mathbb{R}^n$-orbits are Lagrangian submanifolds, we also have $\Psi^*\omega(\partial_{t_i}, \partial_{t_j}) = 0$ for all $1 \leq i, j \leq n$. Therefore it suffices to determine $\Psi^*\omega(\partial_{b_i}, \partial_{t_j})$. By definition,

$$\omega(d\Psi(\partial_{b_i}), d\Psi(\partial_{t_j})) = \omega(d\phi^{\vec{t}}_{\vec{H}} \circ d\sigma(\partial b_i), d\phi^{\vec{t}}_{\vec{H}}(X_{H_j})) = dH_j \circ d\sigma(\partial b_i) = \delta_{ij}$$

using the fact that σ is a section of $\vec{H}$. We therefore conclude that $\Psi^*\omega = \sum_{i=1}^n db_i \wedge dt_i$. The remaining properties of Ψ follow immediately. $\square$

Under this symplectomorphism Ψ we get that $\Lambda^{\vec{H}}|_U = \Psi^{-1}(\sigma(U))$, and hence $\Lambda^{\vec{H}}$ is locally trivial over the submanifold of regular values of $\vec{H}$. However, it's still unclear whether the period over each point $b \in U$ is the same. This can be resolved by upgrading Liouville coordinates to the famous **action-angle coordinates**:

2.19 PROPOSITION (Action-Angle Coordinates). Let $\vec{H}\colon M \to U \subseteq \mathbb{R}^n$ be a completely integrable system over a contractible open set U with only regular and connected fibres, then there is a local change of coordinates $\alpha\colon U \to V \subseteq \mathbb{R}^n$ such that $\alpha \circ \vec{H}$ has standard period lattice $\Lambda^{\alpha\circ H}$ and the map $\Psi\colon U \times \mathbb{R}^n \to M$ descends to a symplectomorphism $U \times \mathbb{R}^k \times (\mathbb{R}/2\pi\mathbb{Z})^{n-k} \to M$.

Proof. As U is contractible, we can find sections $\tau_i\colon U \to \Lambda^{\vec{H}}$ so that $2\pi\tau_1, \cdots, 2\pi\tau_k$ provide a basis of the period lattice $\Lambda^{\vec{H}}$ at each point of U. Let

$$T = \begin{pmatrix} \tau_1^1 & \tau_2^1 & \cdots & \tau_k^1 & 0 & \cdots & 0 \\ \tau_1^2 & \tau_2^2 & \cdots & \tau_k^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tau_1^k & \tau_2^k & \cdots & \tau_k^k & 0 & \cdots & 0 \\ \tau_1^{k+1} & \tau_2^{k+1} & \cdots & \tau_k^{k+1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tau_1^n & \tau_2^n & \cdots & \tau_k^n & 0 & \cdots & 1 \end{pmatrix}$$

be a matrix-valued function on U where the first $k \times n$-entries are given by the sections τ_i and the remaining entries by the identity matrix. Note that $T_b(2\pi\mathbb{Z}^k) = \Lambda_b^{\vec{H}}$, so it suffices to show that T is induced from a change of coordinates $\alpha\colon U \to V$.

Write $\alpha = (\alpha_1, \cdots, \alpha_n)$, then for $\vec{G} = \alpha \circ \vec{H}$ we can compute that

$$d\vec{G} = d\alpha \circ d\vec{H} = \left(\sum_{j=1}^n \frac{\partial \alpha_i}{\partial b_j} db_j \left(dH_j \otimes \frac{\partial}{\partial b_j} \right) \right)_{i=1}^n = \left(\sum_{j=1}^n \frac{\partial \alpha_i}{\partial b_j} dH_j \right)_{i=1}^n ,$$

so by taking the symplectic dual we have

$$X_{G_i} = \sum_{j=1}^n \frac{\partial \alpha_i}{\partial b_j} X_{H_j}$$

for $1 \leq i \leq n$. Write $A = (\partial\alpha_i/\partial b_j)_{i,j}$, then by taking flows and using commutativity we have

$$\phi_{\vec{G}}^t = \phi_{\vec{H}}^{A^T t},$$

so that the period lattices of $\vec{H}$ and $\vec{G}$ are related by

$$A^T \Lambda_b^{\vec{G}} = \Lambda_{\alpha^{-1}(b)}^{\vec{H}}.$$

Now we want $\Lambda^{\vec{G}} = 2\pi\mathbb{Z}^k$ to be the standard period lattice, and from the previous discussion we look for a map α such that

$$\frac{\partial \alpha_i}{\partial b_j} = \begin{cases} \tau_i^j, & \text{if } 1 \leq i \leq k \\ 0, & \text{if } i > k, j \leq k \\ \delta_{ij}, & \text{if } i, j > k \end{cases}$$

As the space U is contractible, such a map α exists iff $\dfrac{\partial \tau_i^j}{\partial b_l} = \dfrac{\partial \tau_i^l}{\partial b_j}$ for all $1 \leq j, l \leq k$ and $1 \leq i \leq n$. Pick a Lagrangian section $\sigma\colon U \to M$ of $\vec{H}$ and use the Liouville coordinate $\Psi\colon V \to U$ so that $\Psi^*\omega = \sum_{i=1}^n db_i \wedge dt_i$ and $\Lambda^{\vec{H}} = \Psi^{-1}(\sigma(U))$. Note that $\Lambda^{\vec{H}} \subseteq \mathbb{R}^n \times U$ is a Lagrangian submanifold that is a cover over U, and is hence a disjoint union classified by the period maps $\tau_i\colon U \to \mathbb{R}^n$ for $1 \leq i \leq k$. The Lagrangian condition then implies that

$$0 = \omega\left(\frac{\partial}{\partial t_j} - \sum_{k=1}^n \frac{\partial \tau_i^j}{\partial b_k}\frac{\partial}{\partial b_k}, \frac{\partial}{\partial t_l} - \sum_{k=1}^n \frac{\partial \tau_i^l}{\partial b_k}\frac{\partial}{\partial b_k}\right) = \frac{\partial \tau_i^l}{\partial b_j} - \frac{\partial \tau_i^j}{\partial b_l},$$

which is exactly the desired condition, and therefore we conclude the proof. $\square$

2.20 Definition. The Liouville coordinates associated to the completely integrable system $\alpha \circ \vec{H}$ is called the **action-angle coordinate**, where $\alpha \circ \vec{H}\colon U \times \mathbb{R}^n \to U$ is the **action coordinate** while the projection $U \times \mathbb{R}^n \to \mathbb{R}^n$ is the **angle coordinate**.

2.21 Corollary (Big Arnold-Liouville Theorem). Assume that $\vec{H}\colon M \to \mathbb{R}^n$ is a completely integrable system with regular and connected fibres, then all the fibres $\mathcal{O}$ are diffeomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$ for some $0 \leq k \leq n$, and there is a neighbourhood of $\mathcal{O}$ in M that is symplectomorphic to $U \times \mathbb{T}^k \times \mathbb{R}^{n-k}$ where the symplectic structure is given by $\sum db_i \wedge dt_i$. Under this symplectomorphism, $\mathcal{O}$ is sent to the fibre $\{b\} \times \mathbb{T}^k \times \mathbb{R}^{n-k}$ for some $b \in U$.

When the fibres are all compact, we get a **Lagrangian torus fibration** $\vec{H}\colon M \to B$ over some subset $B \subseteq \mathbb{R}^n$.

(2e) Hamiltonian G-manifold. An alternative point of view toward the "dimension reduction" is to describe the quotient manifold as a "quotient by a symplectic group action", which we will introduce briefly in this section. Write $\mathrm{symp}(M, \omega)$ to be the space of symplectic vector fields on (M, ω), and $\mathrm{ham}(M, \omega)$ to be the space of hamiltonian vector fields. Directly from the definition 1.8 and the definition of de Rham cohomology we get that

2.22 Lemma. There is a short exact sequence of vector spaces

$$0 \to \mathrm{ham}(M, \omega) \to \mathrm{symp}(M, \omega) \to H^1_{dR}(M; \mathbb{R}) \to 0,$$

and $\mathrm{ham}(M, \omega)$ can be identified with the quotient of space of smooth functions $C^\infty(M, \mathbb{R})$ by the constant functions.

2.23 Definition. Let G be a Lie group. A **symplectic G-action** on a symplectic manifold (M, ω) is a smooth G-action $\rho\colon G \times M \to M$ such that $\rho_g^*\omega = \omega$ for all $g \in G$.

Looking at the infinitesimal action we can see that the Lie algebra $\mathfrak{g}$ of G maps to $\mathrm{symp}(M, \omega)$. Also recall that there is a natural **adjoint action** of G on $\mathfrak{g}$ given by

$$\mathrm{Ad}_g(\xi) \coloneqq \left.\frac{d}{dt}\right|_{t=0} g \exp(t\xi) g^{-1}$$

where $\exp(t\xi)$ is a one-parameter subgroup of G whose derivative at $t = 0$ is the infinitesimal vector field ξ.

2.24 Definition. A symplectic G-action $\rho\colon G \times M \to M$ is **hamiltonian** if furthermore the infinitesimal action $\xi \mapsto X_\xi \in \mathrm{symp}(M, \omega)$ lies in $\mathrm{ham}(M, \omega)$ and admits a linear lift in $C^\infty(M, \mathbb{R})$, and this map is G-equivariant with respect to the adjoint action of G on $\mathfrak{g}$ and the pullback action of G on $C^\infty(M, \mathbb{R})$.

Therefore a Hamiltonian G-action on (M, ω) provides a G-equivariant linear map $\mathfrak{g} \to C^\infty(M, \mathbb{R})$, and note very unrigorously that there is an identification

$$\hom_{\mathbb{R}}(\mathfrak{g}, C^\infty(M, \mathbb{R})) = C^\infty(M, \hom_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})) = C^\infty(M, \mathfrak{g}^\vee),$$

so that a linear map $\mathfrak{g} \to C^\infty(M, \mathbb{R})$ should be identified with a smooth map $\mu_G \colon M \to \mathfrak{g}^\vee$, which we call the **moment map** for the hamiltonian G-action on M. Rigorously, it's defined as follows: for $\xi \in \mathfrak{g}$ and $x \in M$, we define

$$\langle \mu_G(x), \xi \rangle := H_\xi(x),$$

where $H_\xi \in C^\infty(M, \mathbb{R})$ is the image of ξ under the linear lift $\mathfrak{g} \to C^\infty(M, \mathbb{R})$.

2.25 Example. Consider the harmonic oscillator 1.4. In this case the assignment $\xi \mapsto H$ provides the hamiltonian action $\mathfrak{u}(1) \cong \mathbb{R} \to C^\infty(T^*\mathbb{R}, \mathbb{R})$ of the Lie algebra $\mathfrak{u}(1)$ of the unitary group $U(1) \cong \mathbb{S}^1$, which acts as the rotation on the phase space $T^*\mathbb{R} \cong \mathbb{R}^2$ counter-clockwise.

From now on, we assume that $G = T \cong \mathbb{T}^k$ is a torus. Write $\mathfrak{t}^\vee$ to be the dual Lie algebra of T.

2.26 Lemma. Let $c \in \mathfrak{t}^\vee$ be a regular value of the moment map $\mu_T \colon M \to \mathfrak{t}^\vee$, then the level set $\mu_T^{-1}(c)$ is a coisotropic submanifold of M.

Proof. Write the moment map in coordinates, this is essentially Proposition 2.16. □

Note that if $C \subseteq V$ is a coisotropic subspace of a symplectic vector space (V, Ω), then the restriction $\Omega|_C$ has non-trivial kernel $\ker \Omega \subseteq C$ so that Ω descends to the quotient space $\overline{\Omega} \colon \wedge^2 (C/\ker \Omega) \to \mathbb{R}$ which is again non-degenerate. $\ker \Omega$ is often called the **characteristic subspace** of C.

In particular, we know that orbits of T are isotropic submanifolds in $\mu_T^{-1}(c)$, and by dimension count we see that the characteristic subbundle of $T\mu_T^{-1}(c)$ is exactly the tangent bundle of T-orbits. Therefore if the quotient $\mu_T^{-1}(c)/T$ is a smooth manifold, the symplectic structure ω will descend to a symplectic structure on the quotient $\mu_T^{-1}(c)/T$.

Conditions for a quotient space M/G of a smooth manifold M by a Lie group G to be a smooth manifold are well-known [Lee13, Chapter 21]. We will recall some of the terminologies used there.

2.27 Definition. A Lie group action G on M is **free** if the stabilizer subgroup of G at each point of M is trivial, and is **proper** if the map $G \times M \to M \times M$ given by $(g, x) \mapsto (gx, x)$ is proper.

2.28 Theorem ([Lee13, Theorem 21.10]). Suppose G is a Lie group acting smoothly, freely and properly on a smooth manifold M, then the quotient space M/G is a topological manifold of dimension $\dim M - \dim G$ and admits a unique smooth structure such that the quotient map $M \to M/G$ is a smooth submersion.

Now it's an immediate corollary that

2.29 Theorem (Marsden-Weinstein-Meyer). Assume that T acts freely and properly on $\mu_T^{-1}(c)$, i.e. stabilizer subgroup of T at each point of $\mu_T^{-1}(c)$ is trivial and the orbits of T are embedded submanifolds of M, then the quotient space

$$M/\!/_c T = \mu_T^{-1}(c)/T$$

is a symplectic manifold of dimension $\dim M - 2\dim T$.

2.30 EXAMPLE. Consider the symplectic vector space $(\mathbb{C}^{n+1}, \omega_{std})$ regarded as a complex vector space of dimension $2n+2$. Consider a $T = U(1)$-action on $\mathbb{C}^{n+1}$ given in coordinates by

$$\theta.(z_1, \cdots, z_{n+1}) = (e^{i\theta} z_1, \cdots, e^{i\theta} z_{n+1}).$$

One can check that the moment map for T is given by $\mu_T(z_1, \cdots, z_{n+1}) = \frac{1}{2}\sum_{i=1}^{n+1} |z_i|^2$, which is regular over $\mathbb{R}_{>0}$ and has a critical value at 0 whose fibre is exactly one point. For any $c > 0$, $\mu_T^{-1}(c) = \{|z_1|^2 + \cdots + |z_{n+1}|^2 = 2c\}$ defines a sphere of radius $\sqrt{2c}$ in $\mathbb{C}^{n+1}$, and it's clear that the quotient $\mu_T^{-1}(c)/T$ is diffeomorphic to the complex projective space $\mathbb{C}P^n$. However, the symplectic structures on each $\mathbb{C}^{n+1}/\!/_c T$ are different for different $c > 0$: if we rescale $z_i \mapsto \sqrt{c}^{-1} z_i$, then the corresponding symplectic structure scales by c, so that the quotient structure on $\mathbb{C}^{n+1}/\!/_c T$ is c times the quotient symplectic structure on $\mathbb{C}^{n+1}/\!/_1 T$.

2.31 DEFINITION. A group action $G \curvearrowright M$ is **effective** if the action homomorphism $G \to \mathrm{Diff}(M)$ is injective. A symplectic manifold (M, ω) is called **toric** if there is an effective hamiltonian T-action on M with $\dim T = \frac{1}{2}\dim M$.

Note that a basis for $\mu_T \colon M \to \mathfrak{t}^\vee$ provides a completely integrable system on M, and since an effective action has a dense open subset of free orbits, we can find an open subset $U \subseteq \mu_T(M)$ in the image of M under μ_T such that the restriction $\mu_T|_{\mu_T^{-1}(U)} \colon \mu_T^{-1}(U) \to U$ is a smooth Lagrangian torus fibration.

1 PROBLEM. Understand and explain the proof of the following theorem of Atiyah and Guillemin-Sternberg [Ati82, GS82]:

2.32 THEOREM. Let (M, ω) be a compact connected symplectic manifold with an effective hamiltonian T-action, then the image $\mu_T(M)$ of the moment map is a convex polytope in $\mathfrak{t}^\vee$, and the fibres of μ_T are connected.

Reference: [MS17, Theorem 5.5.1], [Aud91, Theorem 4.2.1].

3. Rigidity: Non-squeezing Theorem

In the previous discussions we have shown that symplectic manifolds share some similarities with general smooth topology: there're no local geometries, and their special submanifolds also have pretty good neighbourhood descriptions. Probably non-surprisingly, symplectic manifolds also have some regidities, which distinguishes symplectic topology from smooth topology.

Clearly as $[\omega] \in H^2_{dR}(M;\mathbb{R})$ has to be non-trivial for compact M, the **symplectic volume** $\mathrm{vol}_\omega(M) := \langle [\omega^n], [M] \rangle$ provides a first symplectic invariant that gives some rigidity to symplectic manifolds. For instance, one readily see that

3.1 Lemma. Assume that (M, ω_M) and (N, ω_N) are symplectic manifolds, possibly with boundary, such that $\mathrm{vol}_{\omega_M}(M) > \mathrm{vol}_{\omega_N}(N)$, then there cannot exist any symplectic embedding $M \hookrightarrow N$.

One can therefore ask the following question: is volume the only obstruction to the existence of symplectic embeddings? Let's take a look at the infinitesimal story: given a symplectic vector space $(V, \Omega) = (\mathbb{R}^{2n}, \omega_{\mathrm{std}})$, the space of linear symplectomorphisms of V is the symplectic group $\mathrm{Sp}(V, \Omega)$ of matrices $A\colon V \to V$ such that $A^T \Omega A = \Omega$. As $\wedge^{top}\Omega$ is propotional to the determinant map $\det\colon V \to \mathbb{R}$, the group of and volume-preserving maps is exactly the group of matrices with $\det(A) = 1$, i.e. the special linear group $\mathrm{SL}(V)$.

3.2 Lemma. A matrix of the form

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are $n \times n$ matrices is symplectic if and only if the following conditions hold:

$$A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = \mathrm{Id}\,.$$

In particular, $\mathrm{Sp}(\mathbb{R}^2, \omega_{\mathrm{std}}) = \mathrm{SL}(2, \mathbb{R})$.

So it's clear that in high dimensions volume-preserving is not enough to guarantee the existence of symplectic embeddings. However, we want a non-trivial non-linear example where this can happen. This is the famous theorem of Gromov [Gro85], which we will state and prove in this lecture.

3.3 Theorem (Gromov). Consider the following submanifolds of $\mathbb{C}^n$: for $r > 0$ the open cylinder $Z_R = \mathbb{D}^2_R(0) \times \mathbb{C}^{n-1}$ and the open ball $D_r = \mathbb{D}^{2n}_r(0) \subseteq \mathbb{C}^n$, with inherited symplectic structure from ω_{std}. Then there exists a symplectic embedding $D_r \hookrightarrow Z_R$ if and only if $r \leq R$.

> 13 Exercise. Show that there exists a volume-preserving embedding $D_r \hookrightarrow Z_R$ for any $r, R > 0$. *Hint: Consider linear maps, i.e. matrices.*

The inclusion $D_r \hookrightarrow Z_R$ gives the trivial symplectic embedding provided $r \leq R$, but as $\mathrm{vol}(Z) = \infty$, there're no volume obstructions to the existence of symplectic embeddings $D_r \hookrightarrow Z_R$ when $r > R$. This means we cannot symplectically "squeeze" a ball to embed it into a cylinder of smaller radius, which gives the name **non-squeezing theorem**.

The proof of the non-squeezing theorem is highly non-trivial and it's the first time where the pseudo-holomorphic curve techniques were introduced in symplectic topology.

Although symplectic manifolds does not have local invariants, pseudo-holomorphic curves provide global invariants of symplectic manifolds that was proved extremely effective, and was extensively used later to derive a bunch of symplectic-geometric results. We will go to this later in the lecture.

(3a) Differential calculus on Riemann surfaces. Let's quickly recall some basic facts about Riemann surfaces. By **Riemann surface** we mean a smooth manifold Σ of real dimension 2, and in this lecture we will always assume a Riemann surface is closed, i.e. compact without boundary. An **area form** ω_Σ on Σ is a 2-form that is non-degenerate everywhere, which exists by partition of unity, and as any 2-forms on Σ are closed, it follows that Σ always admits symplectic structures, and therefore also admilts almost complex structures. Moreover, we can compute that the Nijenhuis tensor vanishes, and hence

3.4 Proposition. A Riemann surface Σ admits at least one complex structure j, and hence is a complex manifold.

This can be also shown without using Newlander-Nirenberg theorem. See for example [Che55].

Given an almost complex structure j on Σ, the complexfied tangent bundle $T_{\mathbb{C}}\Sigma := T\Sigma \otimes_{\mathbb{R}} \mathbb{C}$ admits a natural splitting

$$T_{\mathbb{C}}\Sigma = T_{1,0}\Sigma \oplus T_{0,1}\Sigma$$

where $T_{1,0}\Sigma$ and $T_{0,1}\Sigma$ consist of vectors $v \in T_{\mathbb{C}}\Sigma_p$ such that $j_p(v) = iv$ and $j_p(v) = -iv$ respectively. Similarly, the cotangent bundle $T^*_{\mathbb{C}}\Sigma = T^{1,0}\Sigma \oplus T^{0,1}\Sigma$ also splits correspondingly.

3.5 Definition. Under this decomposition, we define

$$\mathcal{A}^{p,q}(\Sigma) := C^\infty(\Sigma, \wedge^p(T^{1,0}\Sigma) \otimes_{\mathbb{C}} \wedge^q(T^{0,1}\Sigma)).$$

In particular, write $\Omega^n(\Sigma) := C^\infty(\Sigma, \wedge^n T^*_{\mathbb{C}}\Sigma)$ to be the space of complex differential forms on Σ, then we get decompositions of vector spaces

$$\Omega^n(\Sigma) \cong \bigoplus_{p+q=n} \mathcal{A}^{p,q}(\Sigma).$$

Of course, $\dim_{\mathbb{R}} \Sigma = 2$, so it's non-trivial only when $n = 0, 1, 2$ and $0 \leq p + q \leq 2$, i.e. we have isomorphisms

$$\Omega^1(\Sigma) \cong \mathcal{A}^{1,0}(\Sigma) \oplus \mathcal{A}^{0,1}(\Sigma),\ \Omega^2(\Sigma) \cong \mathcal{A}^{1,1}(\Sigma)$$

as $\wedge^2(T^{1,0}\Sigma) = \wedge^2(T^{0,1}\Sigma) = 0$.

In local coordinates, let $\mathbb{C} \supseteq U \hookrightarrow \Sigma$ be an open subset of Σ with coordinate $z = x+iy$, then the complexified differential forms are of the form

$$f(x, y)dx + ig(x, y)dy,$$

where $f, g \colon U \to \mathbb{C}$ are smooth complex-valued functions. The standard complex structure $j|_U$ on U is given by $j^*(dx) = -dy$ and $j^*(dy) = dx$, so that $\mathcal{A}^{1,0}(U)$ is spanned by the complex-valued 1-form $dz = dx + idy$ as a module over the algebra $C^\infty(U, \mathbb{C})$, while $\mathcal{A}^{0,1}(U)$ is spanned by the complex-valued 1-form $d\bar{z} = dx - idy$.

Let $\Pi_{p,q}\colon \Omega^{p+q}(\Sigma) \to \Omega^{p,q}(\Sigma)$ be the projection onto the (p,q)-component. The complexified differential $d_{\mathbb{C}} := d \otimes \mathrm{Id}_{\mathbb{C}}$ decomposes with respect to the decomposition of $\Omega^n(\Sigma)$ as $d_{\mathbb{C}}^n = \sum_{p+q=n+1} d^{p,q}$ with

$$d^{p,q} = \Pi_{p,q} \circ d_{\mathbb{C}}^n \colon \Omega^n(\Sigma) \to \Omega^{p,q}(\Sigma).$$

Write $\partial = d^{1,0}$ and $\bar{\partial} = d^{0,1}$.

3.6 Lemma. We have $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$, and ∂ and $\bar{\partial}$ satisfy the Leibniz rule with respect to the wedge product of forms.

Proof. They follow from the direct sum decomposition and the fact that $d_{\mathbb{C}}^2 = 0$ and $d_{\mathbb{C}}$ satisfies the Leibniz rule. □

3.7 Proposition ([Huy05, Proposition 2.6.15]). $d_{\mathbb{C}} = \partial + \bar{\partial}$ if and only if j defines a complex structure on Σ.

14 Exercise. Show that any complex structure j on Σ is compatible with some symplectic structure ω_Σ on Σ. In other words, Σ is a Kähler manifold.

As $\partial^2 = \bar{\partial}^2 = 0$, the sequences

$$0 \to \mathcal{A}^{1,0}(\Sigma) \xrightarrow{\bar{\partial}} \mathcal{A}^{1,1}(\Sigma) \xrightarrow{\bar{\partial}} 0$$

and

$$0 \to \Omega^0(\Sigma) \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1}(\Sigma) \xrightarrow{\bar{\partial}} \mathcal{A}^{0,2}(\Sigma) \xrightarrow{\bar{\partial}} 0$$

are complexes of $\mathbb{C}$-vector spaces, and therefore we can take cohomologies to get **Dolbeault cohomologies**

$$H_{\bar{\partial}}^{p,q}(\Sigma) := \frac{\ker \bar{\partial}|_{\mathcal{A}^{p,q}(\Sigma)}}{\operatorname{Im} \bar{\partial}|_{\mathcal{A}^{p,q-1}(\Sigma)}}.$$

Note that in local coordinates $\bar{\partial} f(z) = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right) d\bar{z}$, hence $\bar{\partial} f = 0$ iff f is holomorphic.

3.8 Definition. A map $f\colon M \to N$ is **holomorphic** if there exists open covers $\{U_i\}$ of M and $\{V_i\}$ of N by coordinate charts such that $f(U_i) \subseteq V_i$ and under these charts $f|_{U_i}\colon U_i \to V_i$ is given by a tuple of holomorphic functions.

f is a **biholomorphism** if f has an inverse $f^{-1}\colon N \to M$ that is also holomorphic. In this case we say M and N are **biholomorphic**. Write $\operatorname{Aut}(M)$ for the group of biholomorphisms of M.

Let's just recall a basic fact from complex analysis:

3.9 Theorem ([JS87, Theorem 2.1.1]). $\operatorname{Aut}(\mathbb{C}P^1) \cong \mathrm{PSL}(2,\mathbb{C})$, where $\mathrm{PSL}(2,\mathbb{C})$ is the quotient of the group $\mathrm{SL}(2,\mathbb{C})$ of invertible matrices with determinant 1 by its center $\{\pm \mathrm{Id}\}$. This isomorphism is explicitly given by the Möbius transformations. In particular, $\dim_{\mathbb{C}} \operatorname{Aut}(\mathbb{C}P^1) = 3$.

Recall from topology that a map $\phi\colon \widetilde{\Sigma} \to \Sigma$ is a **covering map** if fibres $\phi^{-1}(p)$ for each $p \in \Sigma$ is a disjoint union of finitely many points and ϕ is locally trivial, i.e. it induces a fibre bundle over Σ. A generalization to covering map in complex geometry is a **branched cover** $\phi\colon \widetilde{\Sigma} \to \Sigma$, which is a holomorphic map where there exists a finite number of points $p_i \in \Sigma$ such that $\phi^{-1}(\Sigma \setminus \{p_i\}) \to \Sigma \setminus \{p_i\}$ is a covering map, and ϕ is locally given by $z \mapsto z^k$ for some $k > 0$ near each point in $\phi^{-1}(p_i)$.

(3b) Moduli space of pseudo-holomorphic curves. Essential in the proof is to study the space of all pseudo-holomorphic maps $u\colon (\Sigma, j) \to (M, J)$ for some chosen compatible almost complex structure J on a symplectic manifold (M, ω), where (Σ, j) is a Riemann surface with a chosen complex structure j.

3.10 DEFINITION. Let (Σ, j) be a closed Riemann surface with chosen complex structure j and (M, ω) be a symplectic manifold with chosen compatible almost complex structure J. A smooth map $u\colon \Sigma \to M$ is called **J-holomorphic** if it satisfies

$$du \circ j = J \circ du. \tag{3}$$

Equivalently, note that any $u\colon \Sigma \to M$ induces a vector bundle u^*TM over Σ, which is a complex vector bundle with complex structure u^*J. The differential $du\colon T\Sigma \to TM$ can then be understood as a section of the tensor bundle $T^*\Sigma \otimes_{\mathbb{C}} u^*TM$, and using local coordinates we can assume $u\colon \mathbb{C} \supseteq B \to U \subseteq \mathbb{C}^n$ is given by a tuple of smooth complex-valued functions $u = (u_1, \cdots, u_n)$ so that the differential is given by

$$du = \sum_{i=1}^{n} \left(\frac{\partial \operatorname{Re} u_i}{\partial z} \frac{\partial}{\partial x_i} \otimes dz + \frac{\partial \operatorname{Im} u_i}{\partial z} \frac{\partial}{\partial y_i} \otimes dz + \frac{\partial u_i}{\partial \bar{z}} \frac{\partial}{\partial x_i} \otimes d\bar{z} + \frac{\partial \operatorname{Im} u_i}{\partial \bar{z}} \frac{\partial}{\partial y_i} \otimes d\bar{z} \right),$$

so that (3) is equivalent to $\dfrac{\partial u_i}{\partial \bar{z}} = 0$ for all $1 \leq i \leq n$, i.e. $\bar{\partial}_J u = 0$. This description is closer to what we traditionally mean by J-holomorphic.

3.11 DEFINITION. A J-holomorphic curve $u\colon \Sigma \to M$ is **simple** if there does not exist a J-holomorphic curve $v\colon S \to M$ and a branched cover $\phi\colon \Sigma \to S$ such that $u = v \circ \phi$.

An equivalent property for simpleness is the following local description: u is called **somewhere injective** if there exists $z \in \Sigma$ such that $du_z \neq 0$ and $u^{-1}(u(z)) = \{z\}$. This equivalence readily implies that

3.12 PROPOSITION. The set of uninjective points of a simple J-holomorphic curve is discrete. Here uninjective points means $p \in \Sigma$ such that $du_p = 0$ or $u^{-1}(u(p)) \neq \{p\}$.

3.13 DEFINITION. Fix $A \in H_2(M; \mathbb{Z})$, we define the moduli space of J-holomorphic curves of genus g in the homology class A to be

$$\widehat{\mathcal{M}}_g(M, J; A) \coloneqq \{u\colon \Sigma_g \to M | \bar{\partial}_J u = 0,\ u \text{ is simple},\ u_*[\Sigma_g] = A\},$$

where Σ_g is a closed Riemann surface of genus g.

To understand $\widehat{\mathcal{M}}_g(M, J; A)$, the usual way in differential geometry is to find a larger space of maps, for instance we can look at the space of smooth maps $C^\infty(\Sigma_g, M)$, such that $\widehat{\mathcal{M}}_g(M, J; A) \subseteq C^\infty(\Sigma_g, M)$ is a subspace defined by some sections of some vector bundles $E \to C^\infty(\Sigma_g, M)$. In this case, intuitively we can think about $\widehat{\mathcal{M}}_g(M, J; A)$ as the zero locus of the section $\bar{\partial}_J$ of the **Fréchet bundle**

$$C^\infty(\Sigma_g, u^*TM) \to E \to C^\infty(\Sigma_g, M).$$

However, the metric structure on a Fréchet bundle is not good enough to make any meaningful conclusions, so we need to further enlarge the space of maps and sections, to allow non-smooth and even non-continuous maps, in order to “solve” this partial differential equation.

3.14 Definition. A **Banach space** is a vector space V over $\mathbb{R}$ or $\mathbb{C}$ equipped with a norm $\|\cdot\|\colon V \to \mathbb{R}_{\geq 0}$ such that V is complete with respect to the metric induced by the norm. A linear map $f\colon V \to W$ of Banach spaces is **bounded** if there exists $C > 0$ such that $|f(v)| \leq C|v|$ for all $v \in V$.

A **Banach manifold** modelled on a Banach space V is a topological space M with an open cover $\{U_i\}_{i\in I}$ such that there exists embeddings $\phi_i\colon U_i \hookrightarrow V$ such that in the intersections $U_{ij} = U_i \cap U_j$ the transition maps $\phi_j \circ \phi_i^{-1}\colon \phi_i(U_{ij}) \to \phi_j(U_{ij})$ are homeomorphisms.

Note that with the Banach norm it's still possible to define differentials of maps between Banach spaces. Roughly speaking, the **differential** of a continuous map $f\colon V \to W$ at $0 \in V$ is a bounded linear map $df_p\colon V \to W$ such that

$$\lim_{t\to 0} \frac{|f(x+t) - f(x) - df_p(t)|}{\|t\|} = 0.$$

Using this, we are able to define smoothness of maps between Banach spaces, smoothness of Banach manifolds, and smoothness of maps between Banach manifolds. The notion of **Banach bundles** can also be similarly defined, where we require that locally it's homeomorphic to products of Banach spaces, and the transition maps are bounded linear on the fibre component. For more details, see [Lan62, Moo17].

The canonical enlargement of $C^\infty(\Sigma_g, M)$ so that the map spaces have good analysis properties is the so-called Sobolev spaces.

3.15 Definition. Let $(\mathbb{R}^n, \mu)$ be a measure space and $(V, |\cdot|)$ a Banach space. For $k \in \mathbb{Z}_{\geq 0}$ and $p \in [1, \infty)$, the **Sobolev space** $W^{k,p}(\mathbb{R}^n, V)$ is defined to be the completion of $C^\infty(\mathbb{R}^n, V)$ with respect to the norm

$$\|f\|_{W^{k,p}} = \left(\sum_{j=0}^{k} \int_{\mathbb{R}^n} \|d^j f\|^p d\mu\right)^{1/p}.$$

This is a Banach space and in particular when $p = 2$ and the norm in V is induced by an inner product, it's a **Hilbert space** with the inner product given by

$$\langle f, g\rangle_{W^{k,2}} = \sum_{j=0}^{k} \int_{\mathbb{R}^n} \langle d^j f, d^j g\rangle d\mu.$$

As transition maps between coordinate charts of a smooth manifold M is smooth, by introducing a Riemannian metric g on M we can glue local Sobolev spaces to get a **Sobolev manifold** $W^{k,p}(N, M)$ of maps from a smooth manifold N to the Riemannian manifold M. This can be verified to be a C^1-Banach manifold whose tangent space at a point $u \in W^{k,p}(N, M)$ is the Banach space $W^{k,p}(N, u^*TM)$.

For a map $u\colon N \to M$, we can talk about the related topological invariants only when u is continuous, which requires us to look at (k, p) such that $W^{k,p}(\mathbb{R}^n, V) \subseteq C^0(\mathbb{R}^n, V)$. This can not be always achieved, but we have the following Sobolev embedding theorems:

3.16 Theorem ([Eva98, Section 5.6]). *Let U be an open subset of $\mathbb{R}^n$ and $u \in W^{k,p}(U, \mathbb{R})$. If $kp > n$, then $u \in C^0(U, \mathbb{R})$ and there exists $C > 0$ such that $\|u\|_{C^0} \leq C\|u\|_{W^{k,p}}$.*

Back to our situation, we look at Sobolev manifold $W^{1,p}(\Sigma, M)$ where Σ is a compact Riemann surface of genus g and (M, ω) a symplectic manifold with chosen compatible almost complex structure J. When $p > 2$ the above Sobolev embedding theorem implies that $W^{1,p}(\Sigma, M) \subseteq C^0(\Sigma, M)$, so it makes sense to consider the submanifold

$$W^{1,p}(\Sigma, M; A) = \{u\colon W^{1,p}(\Sigma, M) | u_*[\Sigma] = A\}.$$

The tangent space at $u \in W^{1,p}(\Sigma, M; A)$ is then $W^{1,p}(\Sigma, u^*TM)$, and the section we are looking for is $\bar\partial_J \in C^1(W^{1,p}(\Sigma, M; A), TW^{1,p}(\Sigma, M; A) \otimes_{\mathbb{C}} T^{0,1}\Sigma)$, whose zero locus is a priori an enlargement of $\widehat{\mathcal{M}}_g(M, J; A)$. The problem now is to understand the space $\widehat{\mathcal{M}}_g^{1,p}(M, \omega, J; A) \coloneqq \bar\partial_J^{-1}(0)$ inside $W^{1,p}(\Sigma, M; A)$.

(3b-i) Transversality. Recall that given a smooth map $f\colon N \to M$ and a smooth submanifold $A \subseteq M$, we say f is **transversal** to A if for each $x \in f^{-1}(A)$ we have $T_{f(x)}M = T_{f(x)}A + df_x(T_xN)$. When f is transversal to A the preimage $f^{-1}(A)$ is a smooth submanifold of N with codimension equal to the codimension of A in M.

Now let $E \to M$ be a smooth finite-rank vector bundle over M, we know that the restriction $TE|_M \cong TM \oplus E$ admits a natural splitting into the tangent bundle TM of M and the vector bundle E. Let $s\colon M \to E$ be a section of E, then the transversality condition for $s^{-1}(0) \subseteq M$ to be a smooth submanifold of M of codimension $r = \operatorname{rank} E$ is equivalent to the projection $ds|_{s^{-1}(0)}\colon TM \to TE \twoheadrightarrow E$ being surjective for all $x \in s^{-1}(0)$.

In inifnite dimensions, in addition to the bundle map $TM \to E$ being surjective, we need to further require that the surjection admits a right inverse $E \to TM$, i.e. the composition $E \to TM \xrightarrow{ds} E$ is the identity map. This splitting provides a direct sum decomposition $TM \cong K \oplus E$ where $K = \ker(ds)|_{s^{-1}(0)}$ is the tangent bundle of the smooth manifold $s^{-1}(0)$.

For $\mathcal{E}^{1,p} = TW^{1,p}(\Sigma, M; A) \otimes_{\mathbb{C}} T^{0,1}\Sigma$ the aformentioned Banach bundle over $\mathcal{B}^{1,p} = W^{1,p}(\Sigma, M; A)$, we can compute the differential of $\bar\partial_J$ at $u \in W^{1,p}(\Sigma, M; A)$, which is also often called the **linearization** of $\bar\partial_J$ at u such that $\bar\partial_J(u) = 0$, as follows: recall that $TE|_M \cong TM \oplus E$ admits a splitting into TM and $\mathcal{E}^p$, where $d\bar\partial_J$ is identity on the first component, so it suffices to determine the projection $D\bar\partial_J\colon T\mathcal{B}^{1,p} \to \mathcal{E}^p$, whose restriction to $u \in \mathcal{B}^{1,p}$ is given by an operator $D_u\colon W^{1,p}(\Sigma, u^*TM) \to W^{1,p}(\Sigma, u^*TM \otimes T^{0,1}\Sigma)$.

3.17 Definition. J is called **regular** for A and Σ if D_u is split surjective for every u with $\bar\partial_J u = 0$. We denote by

$$\mathcal{J}_c^{reg}(M, \omega; \Sigma, A)$$

the space of all compatible almost complex structures J for (M, ω) that is furthermore regular.

We will sketch but omit most details of the proof of the following theorem:

3.18 Theorem ([MS12, Theorem 3.1.6]). Fix a compact Riemann surface $(\Sigma, j_\Sigma, d\operatorname{vol}_\Sigma)$ and a homology class $A \in H_2(M; \mathbb{Z})$.

(i) If $J \in \mathcal{J}_c^{reg}(M, \omega; \Sigma, A)$ then the space $\widehat{\mathcal{M}}_g^{1,p}(M, \omega, J; A)$ is a smooth manifold of dimension

$$\dim \widehat{\mathcal{M}}_g^{1,p}(M, \omega, J; A) = n(2 - 2g) + 2c_1(A)$$

carrying a natural orientation.

(ii) The set $\mathcal{J}_c^{reg}(M, \omega; \Sigma, A)$ is **residual** in $\mathcal{J}_c(M, \omega)$, i.e. it contains a countable intersection of open dense subsets of $\mathcal{J}$.

For a complex vector bundle $E \to M$ over some smooth manifold M, we can define its **determinant line bundle** $\det E = \wedge^r E \to M$, where r is the rank of E. For each map $u\colon \Sigma \to M$, we have the natural pull-back bundle $u^* \det E \to \Sigma$ which defines a complex line bundle over Σ. We define the element $c_1(A)$ to be the signed count of the points in a generic section of $u^* \det TM$. We can show that this does not depend on the choice of sections.

Sketch of Proof. Instead of studying D_u for fixed almost complex structure J, we consider the **universal moduli space**

$$\mathcal{U}^{k,p,\ell}(M;A) = \mathcal{B}^{k,p}(M;A) \times \mathcal{J}^\ell_\omega(M)$$

where $\mathcal{J}^\ell_\omega(M)$ is the completion of $\mathcal{J}_c(M,\omega)$ under the C^ℓ-metric. We are looking in this space at the family of moduli spaces $\widehat{\mathcal{M}}^{k,p,\ell}_g(M,\omega,\mathcal{J}^\ell_c;A) \xrightarrow{\pi} \mathcal{J}^\ell_c(M,\omega)$ parametrized by $\mathcal{J}^\ell_\omega(M)$. To prove (ii), it suffices for us to show that for generic choice of $J \in \mathcal{J}_c(M,\omega) \subseteq \mathcal{J}^\ell_\omega(M)$, the corresponding operator $D_{u,J}$ is split surjective for every $u \in \widehat{\mathcal{M}}^{k,p,\ell}_g(M,\omega,J;A)$.

An operator $D\colon V \to W$ between Banach spaces V and W is **Fredholm** if $\dim\ker D$ and $\dim\operatorname{coker} D$ are all finite-dimensional. When D is Fredholm, we define its **Fredholm index** by $\operatorname{Ind}(D) \coloneqq \dim\ker D - \dim\operatorname{coker} D$.

For maps $f\colon X \to Y$ between separable C^1-Banach manifolds with Fredholm differentials, Smale [Sma65] showed that the set $y \in Y^{reg}$ of values in Y where df is surjective is residual, which generalizes Sard's theorem to infinite dimensions.

In particular, we are now considering sections of the vector bundle $\mathcal{E}^{k-1,p} \to \mathcal{U}^{k,\ell,p}$ where the fibre at u is given by $W^{k-1,p}(\Sigma, u^*TM \otimes T^{1,0}\Sigma)$. The space $\mathcal{J}^\ell_\omega(M)$ is Banach with tangent space at J provided by bundle endomorphisms $Y\colon TM \to TM$ satisfying $YJ + JY = 0$ and $\omega(Yv,w) + \omega(v,Yw) = 0$ for all $q \in M$ and $v,w \in T_qM$. Write $s\colon \mathcal{U}^{k,\ell,p} \to \mathcal{E}^{k-1,p}$ to be the section, then we can compute that

$$Ds_{(u,J)}(\xi, Y) = D_{u,J}(\xi) + \frac{1}{2} Y(u) \circ du \circ j.$$

3.19 Proposition (Fredholm Property for Elliptic Operators). Fix a homology class $A \in H_2(M;\mathbb{Z})$, an integer $\ell \geq 2$, a real number $p > 2$, and an integer $1 \leq k \leq \ell$. Then the following holds.

(i) The operator $D_{u,J}$ is Fredholm with indices $\operatorname{Ind} D_{u,J} = n(2-2g) + 2c_1(A)$.

(ii) $Ds_{(u,J)}$ is surjective whenever $s(u,J) = 0$ and u is **somewhere injective**, i.e. there is $p \in \Sigma$ and $\delta > 0$ such that $d_M(u(p), u(\zeta)) \geq \delta d_\Sigma(z,\zeta)$ for all $\zeta \in \Sigma$.

(iii) $d\pi\colon (\xi, Y) \mapsto Y$ is Fredholm.

This Proposition then implies the theorem. $\square$

Moreover, we can show that $\widehat{\mathcal{M}}^{k,p}_g(M,\omega,J;A)$ does not depend on the choice of the pair (k,p), by the following elliptic regularity result:

3.20 Proposition (Elliptic Regularity). Assume J is a smooth ω-compatible almost complex structure and $u\colon \Sigma \to M$ a J-holomorphic curve of class $W^{1,p}$ with $p > 2$, then u is smooth.

In particular, all maps in $\widehat{\mathcal{M}}_g^{k,p}(M,\omega,J;A)$ are smooth, and hence we can remove the superscripts and simply write $\widehat{\mathcal{M}}_g(M,\omega,J;A)$. This is a smooth manifold of dimension $n(2-2g)+2c_1(A)$. We mostly focus on the case when $g=0$, so that $\widehat{\mathcal{M}}_0(M,\omega,J;A)$ is a smooth manifold of dimension $2n+2c_1(A)$.

(3b-ii) Compactness. For the proof of the non-squeezing theorem, we need to consider the moduli space of curves with a marked point

$$\widehat{\mathcal{M}}_{g,1}(M,\omega,J;A)=\{(u,z)|u\in\widehat{\mathcal{M}}_0(M,\omega,J;A), z\in\Sigma\}\cong\widehat{\mathcal{M}}_g(M,\omega,J;A)\times\Sigma,$$

which is still a smooth manifold of dimension $n(2-2g)+2c_1(A)+2$. By taking the image of u under the given point z, we get an evaluation map

$$\mathrm{ev}\colon \widehat{\mathcal{M}}_{g,1}(M,\omega,J;A)\to M.$$

Note that $\mathrm{PGL}(2,\mathbb{C})$ acts on $\widehat{\mathcal{M}}_{g,1}(M,\omega,J;A)$ by reparametrization of the domain Σ, and hence does not affect the image of u. We want to consider the **unparametrized** version of the mouduli space

$$\mathcal{M}_{g,1}(M,\omega,J;A)\coloneqq\widehat{\mathcal{M}}_{g,1}(M,\omega,J;A)/\operatorname{PGL}(2,\mathbb{C}),$$

where elements in $\mathcal{M}_{g,1}(M,\omega,J;A)$ are just curves in M independent of the parametrization. As the image of the evaluation map does not change under the $\mathrm{PGL}(2,\mathbb{C})$-action, there is still a natural evaluation map $\mathrm{ev}\colon\mathcal{M}_{g,1}(M,\omega,J;A)\to M$, which can also be shown to be smooth.

Let $[D]\in H_*(M;\mathbb{Z})$ be a cycle represented by a closed subset D in M, then the preimage $\mathrm{ev}^{-1}(D)\subseteq\mathcal{M}_{g,1}(M,\omega,J;A)$ is the set of pairs (u,z) such that $u(z)\in D$. To get a better analysis of the preimage $\mathrm{ev}^{-1}(D)$, it'll be better if the moduli space $\mathcal{M}_{g,1}(M,\omega,J;A)$ is compact. However, $\mathcal{M}_{g,1}(M,\omega,J;A)$ is not compact in general, but geometrically we can provide a natural compactification of $\mathcal{M}_{g,1}(M,\omega,J;A)$ by adding the moduli space of, **cusp-curves** in the ancient language, or equivalently **stable curves** after Kontsevich [Kon95].

Let Σ be a compact Riemann surface of genus g with m marked points $z_1,\cdots,z_m$ and embedded disjoint simple closed curves $\gamma_1,\cdots,\gamma_k$ that do not pass through any marked points. A **cusp-curve** is the quotient space

$$\overline{\Sigma}\coloneqq\Sigma/\{x\sim y\Leftrightarrow\text{there exists a contractible curve }\gamma_i\text{ so that }x,y\in\gamma_i\}$$

given by shrinking each γ_i to a point.

3.21 Theorem (Uhlenbeck-Gromov Compactness). Let $[u_i]\in\mathcal{M}_{g,m}(M,\omega,J;A)$ be a sequence of J-holomorphic curves with m marked points in the homology class A in a compact symplectic manifold (M,ω). Then there exists a subsequence $\{[u_{i_j}]\}$ that converges to a cusp-curve $u_\infty\colon\overline{\Sigma}\to M$, in an appropriate sense.

This theorem is intuitively easy to understand, while in practice difficult to formulate. A rigorous treatment can be found in [MS12, Chapters 4 and 5] via Gromov topology and stable maps. The rough idea is the following: firstly note that the **energy**

$$E(u)=\int_\Sigma u^*\omega=\omega(A)=\int_\Sigma|du|^2d\operatorname{vol}_\Sigma$$

of $u \in \mathcal{M}_{g,m}(M, \omega, J; A)$ is uniformly bounded above. If furthermore $|du_i|^2$ is uniformly bounded over Σ, by compactness of M and the folowing theorem, $\{u_i\}$ has a subsequence that C^1, and after taking care of the regularity, in C^∞, to a J-holomorphic curve $u_\infty \colon \Sigma \to M$.

3.22 Theorem (Arzelà-Ascoli). Let Σ be a compact Hausdorff space and M a metric space. Assume that $\mathcal{F} \subseteq C(\Sigma, M)$ is a family of continuous maps. Then $\mathcal{F}$ is relatively compact in the compact-open topology of $C(\Sigma, M)$ if and only if it is

(i) (Equicontinuous) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_M(f(z), f(\zeta)) < \varepsilon$ for all $f \in \mathcal{F}$ and $z, \zeta \in \Sigma$ with $d_\Sigma(z, \zeta) < \delta$.

(ii) (Pointwise bounded) For every $z \in \Sigma$ the set $\{f(z) | f \in \mathcal{F}\}$ is bounded in M.

Therefore the problem only arises at points $z \in \Sigma$ where $|du_i(z)|^2$ is unbounded. For each such point, we can normalize the norm $|du_i|^2$ in a neighbourhood of z by rescaling, which looks like "bubbling off" the neighbourhood of z from Σ, and if we pass to the limit, we will gradually see a copy of $\mathbb{C}$ with z as the origin, where the theorem of Arzelà-Ascoli can be applied locally to provide a local C^∞-convergence to a J-holomorphic curve $u_\infty \colon \mathbb{C} \to M$ up to passing to subsequences. We can then show that u_∞ naturally extends to a map from $\mathbb{C}P^1$ to M, and this extended point is in the image of the limit of the remaining parts of the curve.

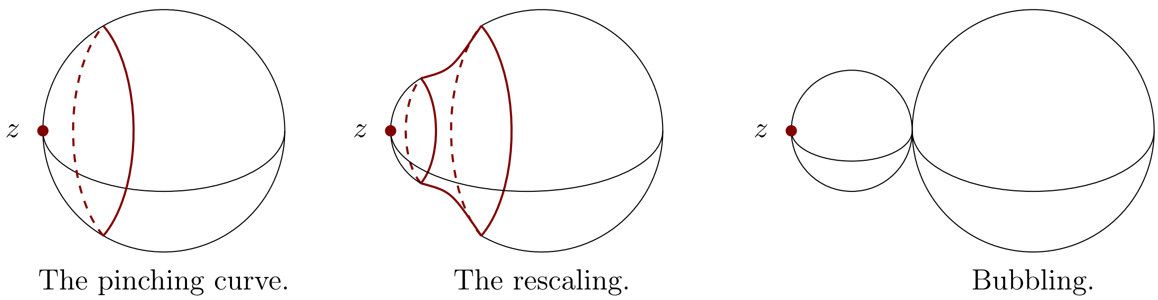

Figure 2: The bubbling procedure.

We write $\overline{\mathcal{M}}_{g,m}(M, \omega, J; A)$ to be the compactification of $\mathcal{M}_{g,m}(M, \omega, J; A)$ by adding all the possible limits of sequences of J-holomorphic curves. From Uhlenbeck-Gromov compactness, there will be two possible types of compactifications: cusp-curves and non-simple curves.

Suppose $g = 0$, then the cusp-curves are unions of $\mathbb{P}^1$s along several points, and if we represent each sphere by a vertex, and connect two vertices by an edge if the two spheres intersect at a point. When $g = 0$, cusp-curves that are limits of some holomorphic spheres are always represented by **trees**, i.e. graphs with no cycles, and we can compute the dimension of these moduli space of cusp-curves as follows. Regard intersection points as marked points on a sphere, so that we can write the corresponding moduli space by

$$\mathcal{M}_{0,m_i,e_i}(M, \omega, J; A_i)$$

where m_i is the number of marked points on this component, $\sum A_i = A$ and e_i is the number of points that intersect with other components. Let $T = (V, E, W)$ be the tree where V is the set of vertices, E is the set of edges and $W \colon V \to \mathbb{Z}_{\geq 0}$ assigns to each vertex the number of marked points on the corresponding component, then the moduli space $\mathcal{M}_T(M,\omega,J;A)$ of cusp-curves corresponding to T can be identified with

$$\mathcal{M}_T(M,\omega,J;A) \cong \left\{ (u_i)_{i\in V} \in \prod_{i\in V} \mathcal{M}_{0,W(i),e_i}(M,\omega,J;A_i) \middle| \substack{\mathrm{ev}_{e,i}(u_i)=\mathrm{ev}_{e,j}(u_j) \text{ whenever} \\ e\in E \text{ is an edge connecting } i \text{ and } j.} \right\}$$

Up to further deforming J we can assume that the evaluation map $\mathrm{ev}_{e,i}$ is transversal to $\mathrm{ev}_{e,j}$, so that the dimension of $\mathcal{M}_T(M,\omega,J;A)$ is given by

$$\dim \mathcal{M}_T(M,\omega,J;A) = 2n|V| + 2m - 6|V| - 2(n-2)|E| + 2c_1(A).$$

As $|E| = |V| - 1$ for a tree, we have $\dim \mathcal{M}_T(M,\omega,J;A) = 2n + 2m - 4 + 2c_1(A) - 2|V|$. In particular, all non-trivial cusp-curves have codimension at least 2 in $\overline{\mathcal{M}}_{0,m}(M,\omega,J;A)$.

For non-simple curves appearing in the boundary of $\overline{\mathcal{M}}_{0,m}(M,\omega,J;A)$, one can replace it by simple ones: if $A = \sum_\alpha m_\alpha A_\alpha$ is a decomposition into some other curve classes, we can find simple cusp-curves in $\overline{\mathcal{M}}_{0,m}(M,\omega,J;\sum A_\alpha)$ having the same image as the non-simple ones. Note that this moduli space should have dimension

$$2n + 2m - 4 + 2\sum_\alpha c_1(A_\alpha) - 2|V|,$$

which has codimension ≥ 2 in $\overline{\mathcal{M}}_{0,m}(M,\omega,J;A)$ only when $c_1(A_\alpha) \geq 0$ for all α. This will hold if we impose the following condition on M:

3.23 Definition. A symplectic manifold (M,ω) is **semipositive** if for every $A \in H_2(M;\mathbb{Z})$,

$$\omega(A) > 0, \quad c_1(A) \geq 3 - n \Rightarrow c_1(A) \geq 0.$$

Note that when $c_1(A) < 3-n$, the moduli space $\mathcal{M}_{0,0}(M,\omega,J;A)$ is empty, and hence this condition is enough to ensure the abovementioned codimension condition. Compactifying strata having codimension 2 implies that the fundamental cycle $\left[\overline{\mathcal{M}}_{0,m}(M,\omega,J;A)\right]^{vir} \in H_{2n+2m-6+2c_1(A)}\left(\overline{\mathcal{M}}_{0,m}(M,\omega,J;A);\mathbb{Z}\right)$ is well-defined.

(3c) Proof of the non-squeezing theorem. With the moduli theory of J-holomorphic curves established, we are ready to prove the non-squeezing theorem. Let's start with a lower bound for area of J-holomorphic curves in an Euclidean space.

(3c-i) Isoperimetric inequality for J-holomorphic curves. Let $\gamma\colon \mathbb{S}^1 \to V$ be a smooth loop in a symplectic vector space $(V = \mathbb{C}^n, \omega_{\mathrm{std}}, J_{\mathrm{std}})$, then we can define the **symplectic action** of γ by $\mathcal{A}(\gamma) = \frac{1}{2}\int_0^{2\pi} \omega(\dot\gamma(\theta),\gamma(\theta))d\theta$, the **length** of γ by $L(\gamma) := \int_0^{2\pi} |\dot\gamma(\theta)|d\theta$, and the **energy** of γ by $E(\gamma) := \frac{1}{2}\int_0^{2\pi} |\dot\gamma(\theta)|^2 d\theta$.

3.24 Lemma ([MS12, Lemma 4.4.4]). Under the above hypotheses, we have the inequality

$$|\mathcal{A}(\gamma)| \leq \frac{1}{4\pi} L(\gamma)^2.$$

Proof. Using the vector version of Euler's identity, we have $e^{k\theta J} = \cos(k\theta)\,\mathrm{Id} + \sin(k\theta)J$, so that for given vectors $v, w \in V$ we have

$$\begin{aligned}\omega(e^{k\theta J}v, e^{m\theta J}w) &= \cos(k\theta)\cos(m\theta)\omega(v,w) + \sin(k\theta)\sin(m\theta)\omega(v,w) \\ &\quad + \cos(k\theta)\sin(m\theta)\omega(v,Jw) + \sin(k\theta)\cos(m\theta)\omega(Jv,w) \\ &= \cos((k-m)\theta)\omega(v,w) + \sin((k-m)\theta)\omega(v,Jw).\end{aligned}$$

Integrate θ over the interval $[0, 2\pi]$ we get

$$\int_0^{2\pi} \omega(e^{k\theta J}v, e^{m\theta J}w)d\theta = \begin{cases} 2\pi\omega(v,w), & k = m, \\ 0, & \text{otherwise.}\end{cases}$$

For any smooth $\gamma\colon \mathbb{S}^1 \to V$, consider its Fourier series expansion

$$\gamma(\theta) = \sum_{k=0}^{\infty} e^{k\theta J}v_k,$$

so that $\dot\gamma(\theta) = \sum_{k=0}^{\infty} k e^{k\theta J} J v_k$. We can then compute that

$$|\mathcal{A}(\gamma)| = \frac{1}{2}\left|\int_0^{2\pi}\omega(\dot\gamma(\theta),\gamma(\theta))d\theta\right| = \sum_{k=0}^{\infty} k\pi|v_k|^2 \le \sum_{k=0}^{\infty} k^2\pi|v_k|^2 = E(\gamma).$$

Suppose that γ is immersed, then we can reparametrize γ by its arc-length so that $|\dot\gamma(\theta)| = L(\theta)/2\pi$, then we can compute that

$$E(\gamma) = \frac{1}{2}\int_0^{2\pi}|\dot\gamma(\theta)|^2 d\theta = \frac{1}{2}\int_0^{2\pi}\frac{L(\gamma)^2}{4\pi^2}d\theta = \frac{1}{4\pi}L(\gamma)^2.$$

This finishes the proof whenever γ is immersed. For general γ, choose a sequence $\{a_\nu\}_\nu$ so that $a_\nu \to 0$ as $\nu \to \infty$ and $a_\nu \ne e^{-\theta J}J\dot\gamma(\theta)$ for each ν, and define $\gamma_\nu(\theta) = \gamma(\theta) + e^{\theta J}a_\nu$, then γ_ν is immersed for each ν with $\gamma_\nu \rightrightarrows \gamma$ C^∞-uniformly, and hence $\mathcal{A}(\gamma_\nu) \to \mathcal{A}(\gamma)$ and $L(\gamma_\nu) \to L(\gamma)$ as $\nu \to \infty$, so that the inequality also holds for general γ. $\square$

3.25 Corollary. Let $u\colon S \to V$ be a J-holomorphic curve with $u(0) = 0$ where S is a not necessarily compact Riemann surface, then for each $\varepsilon > 0$ we have

$$\int_{S\cap B_\varepsilon(0)} u^*\omega \ge \pi\varepsilon^2.$$

Proof. Let

$$A(\varepsilon) = \int_{S\cap B_\varepsilon(0)} u^*\omega = \int_0^{2\pi}\omega\left(du(V), du(jV)\right)d\theta$$

be the symplectic area of u in the ball $B_\varepsilon(0)$, where V is a unit vector field on S. We can choose V such that near $\partial(S\cap B_\varepsilon(0))$ the vector $du(V)$ is the projection of the radial vector field $r\partial/\partial r$ on $\mathbb{C}^n$ to the tangent space of $u(S)$, so that $A(\varepsilon) = \mathcal{A}(\partial(S\cap B_\varepsilon(0)))$. On the other hand, coarea formula implies that

$$\frac{d}{d\varepsilon}A(\varepsilon) = \int_0^{2\pi}\frac{\omega(du(V), du(jV))}{|du(V)|}d\theta \ge \int_0^{2\pi}\frac{\omega(\dot\gamma_\varepsilon(\theta),\gamma_\varepsilon(\theta))}{|\dot\gamma_\varepsilon(\theta)|}d\theta = L(\gamma_\varepsilon).$$

Therefore we have

$$\frac{\dot{A}(\varepsilon)}{\sqrt{A(\varepsilon)}} \geq \frac{L(\gamma_\varepsilon)}{\sqrt{A(\varepsilon)}} \geq 2\sqrt{\pi}.$$

By taking integration with respect to ε we get that $A(\varepsilon) \geq \pi\varepsilon^2$ for each $\varepsilon > 0$ such that $S \cap \partial B_\varepsilon(0) \neq \emptyset$. □

For general not necessarily linear almost complex structures J in $\mathbb{C}^n$, we cannot achieve isoperimetric inequality with such sharp constant, but there is still a weaker version, which we will introduce here without a proof.

3.26 Lemma (Monotonicity Lemma). Let $u\colon \Sigma \to M$ be a J-holomorphic curve in an almost Kähler manifold (M, ω, J), then there are constants c_0 and $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$ and if $x \in \Sigma$ is such that $u(\Sigma) \cap \overline{B}_\varepsilon(u(x))$ is compact with boundary contained in the boundary $\partial B_\varepsilon(u(x))$

$$A(S \cap \overline{B}_\varepsilon(u(x))) \geq \frac{\pi}{1 + c_0\varepsilon}\varepsilon^2.$$

(3c-ii) Holomorphic spheres in aspherical manifolds. For the sake of applications, we only look at a very special type of compact symplectic manifolds where the moduli space of holomorphic curves behave very nicely.

3.27 Definition. A compact symplectic manifold (M, ω) is called **symplectically aspherical** if $\omega(A) = 0$ for all $A \in H_2(M; \mathbb{Z})$ that can be represented by a J-holomorphic sphere $u\colon \mathbb{S}^2 \to M$ for some ω-compatible almost complex structure J.

The monotonicity lemma then implies that

3.28 Corollary. For symplectically aspherical manifolds (M, ω, J), there does not exist non-constant J-holomorphic spheres $u\colon \mathbb{S}^2 \to M$.

3.29 Example. Let $M = \mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be a $2n$-dimensional torus with symplectic structure $\omega = \sum_{i=1}^{n} d\theta_i \wedge d\varphi_i$ where θ_i and φ_i are the angle coordinates in $\mathbb{R}^{2n}$. Then we can verify that M is symplectically aspherical as follows. Note that $q\colon \mathbb{R}^{2n} \to M$ is a universal cover of M, and since $\mathbb{S}^2$ is simply connected, any smooth map $u\colon \mathbb{S}^2 \to M$ can be lifted to a smooth map $\tilde{u}\colon \mathbb{S}^2 \to \mathbb{R}^{2n}$. Now we must have $u_*[\mathbb{S}^2] = q_*\tilde{u}_*[\mathbb{S}^2] = 0$ as $H^2(\mathbb{R}^{2n}; \mathbb{Z}) = 0$, so we conclude that $\omega(u_*[\mathbb{S}^2]) = 0$.

On the other hand, for J-holomorphic curves $u\colon \mathbb{S}^2 \to (\mathbb{P}^1, J_{\mathrm{std}})$, we can show using Riemann mapping theorem that any non-constant holomorphic curve u has to be a multiple cover of $\mathbb{P}^1$, and therefore simple ones has to be biholomorphisms. This implies that $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^1, \omega_{\mathrm{FS}}, J_{\mathrm{std}}; A) \cong \mathbb{P}^1$ is already compact.

Recall the open charts of $\mathbb{P}^1$ in Example 2.2: we have $U_0 = \{[z_0 : z_1] \in \mathbb{P}^1 | z_0 \neq 0\} \cong \mathbb{C}_z$ and $U_1 = \{[z_0 : z_1] \in \mathbb{P}^1 | z_1 \neq 0\} \cong \mathbb{C}_w$, where the transition map is given by $w = 1/z = \bar{z}/|z|^2$. We consider the holomorphic tangent bundle $T\mathbb{P}^1$ of $\mathbb{P}^1$, whose restriction to U_0 is given by $\mathbb{C}_z \times \mathbb{C}\partial/\partial z$ and to U_1 is given by $\mathbb{C}_w \times \mathbb{C}\partial/\partial w$. The transition map of $T\mathbb{P}^1$ can be computed as follows:

$$w_*\left(\frac{\partial}{\partial z}\right) = \frac{dw}{dz}\frac{\partial}{\partial w} = -\frac{1}{z^2}\frac{\partial}{\partial w} = -w^2\frac{\partial}{\partial w}.$$

In particular, we can find a section $s\colon \mathbb{P}^1 \to T\mathbb{P}^1$ defined by $s(z) = z\partial/\partial z$ on U_0 and $s(w) = -w\partial/\partial w$ on U_1, which has two zeroes 0 and ∞, both non-degenerate. Therefore

s is a section of $T\mathbb{P}^1$ transverse to the zero section with exactly two zeroes, which shows that $c_1(A) = 2$ where $A = [\mathrm{pt}]$ is the point class of $\mathbb{P}^1$ generating $H_0(\mathbb{P}^1;\mathbb{Z})$. By dimension formula, we have

$$\dim \mathcal{M}_{0,1}(\mathbb{P}^1, \omega_{\mathrm{FS}}, J_{\mathrm{std}}; A) = 2 + 2 - 6 + 2c_1(A) = 2,$$

and the previous discussion implies that the fibre $\mathrm{ev}^{-1}(p)$ of the evaluation map

$$\mathrm{ev}\colon \overline{\mathcal{M}}_{0,1}(\mathbb{P}^1, \omega_{\mathrm{FS}}, J_{\mathrm{std}}; A) \to \mathbb{P}^1$$

consists of exactly one point, and hence represents the class $[\mathrm{pt}] \in H_0(\overline{\mathcal{M}}(\mathbb{P}^1, \omega_{FS}, J_{\mathrm{std}});\mathbb{Z})$.

3.30 Proposition. Let $M = \mathbb{P}^1 \times N$ be a product of $\mathbb{P}^1$ and a symplectically aspherical manifold N, where the symplectic structure is given by $\omega_M = \pi_1^*\omega_{\mathbb{P}^1} + \pi_2^*\omega_N$. Then for any regular ω_M-compatible almost complex structure $J \in \mathcal{J}^{reg}_{\omega_M}(M)$ and any point $p \in M$, there exists at least one simple J-holomorphic sphere passing through p.

Proof. It suffices to show that the class $[\mathrm{ev}_J^{-1}(p)]$ does not vanish in

$$H_*\left(\overline{\mathcal{M}}_{0,1}(M, \omega_M, J; [\mathbb{P}^1 \times \{\mathrm{pt}\}]);\mathbb{Z}\right)$$

for each regular J. Any two compatible almost complex structures on M can be connected by a smooth path $\{J_t\}_{t\in[0,1]}$ of ω_M-compatible almost complex structures, though this family might pass through some non-regular almost complex structures. Nontheless, we can consider the moduli space of $\{J_t\}_{t\in[0,1]}$-holomorphic curves

$$\overline{\mathcal{M}}_{0,1}(M, \omega_M, \{J_t\}; [\mathbb{P}^1 \times \{\mathrm{pt}\}]) = \{(t,[u]) | t \in [0,1], [u] \in \overline{\mathcal{M}}_{0,1}(M, \omega_M, J_t; [\mathbb{P}^1 \times \{\mathrm{pt}\}])\},$$

which by forgetting $[u]$ admits a natural projection to $[0,1]_t$.

3.31 Definition. Fix a compact Riemann surface $(\Sigma, j, d\,\mathrm{vol}_\Sigma)$ and a homology class $A \in H_2(M;\mathbb{Z})$. Let $J_0, J_1 \in \mathcal{J}^{reg}_{\omega_M}(M)$ be two regular ω_M-compatible almost complex structures. A homotopy $[0,1] \to \mathcal{J}_{\omega_M}(M)$, $\lambda \mapsto J_\lambda$ from J_0 to J_1 is called **regular** (for A and Σ) if

$$\Omega^{0,1}(\Sigma, u^*TM) = \operatorname{im} D_{J_\lambda,u} + \mathbb{R}v_\lambda$$

for every $(\lambda, u) \in \mathcal{M}_{g,0}(M, \omega_M, \{J_\lambda\}; A)$, where $v_\lambda = (\partial_\lambda J_\lambda) du \circ j_\Sigma$ is the image in $\Omega^{0,1}(\Sigma, u^*TM)$ of the tangent vector to the path $\{J_\lambda\}$. We write $\mathcal{J}^{reg}_{\omega_M}(M, A; J_0, J_1)$ for the space of all regular homotopies.

3.32 Proposition (Regularity in families). Let $J_0, J_1 \in \mathcal{J}^{reg}_{\omega_M}(M)$ be two regular ω_M-compatible almost complex structures. Fix a compact Riemann surface $(\Sigma, j, d\,\mathrm{vol}_\Sigma)$ and $A \in H_2(M;\mathbb{Z})$.

(i) If $\{J_\lambda\}_{\lambda\in[0,1]}$ is regular then $\mathcal{M}_{g,0}(M, \omega_M, \{J_\lambda\}; A)$ is a smooth orientable manifold with boundary $\mathcal{M}_{g,0}(M, \omega_M, J_0; A) \sqcup \mathcal{M}_{g,0}(M, \omega_M, J_1; A)$. The boundary orientation agrees with the orientation of $\mathcal{M}_{g,0}(M, \omega_M, J_1; A)$ and is opposite to the orientation of $\mathcal{M}_{g,0}(M, \omega_M, J_0; A)$.

(ii) The set $\mathcal{J}^{reg}_{\omega_M}(M, A; J_0, J_1)$ is residual in the space of all smooth homotopies from J_0 to J_1.

The codimension 2 property of moduli spaces of cusp-curves for each J_i implies that the cobordism relation holds after passing to the compactification. Moreover, by further restricting to residual subsets we can assume that the evaluation map

$$\mathrm{ev}\colon \overline{\mathcal{M}}_{0,1}(M, \omega_M, \{J_\lambda\}; [\mathbb{P}^1 \times \{\mathrm{pt}\}]) \to M$$

is transversal to p, so that $\mathrm{ev}^{-1}(p) \in \overline{\mathcal{M}}_{0,1}(M, \omega_M, \{J_\lambda\}; [\mathbb{P}^1 \times \{\mathrm{pt}\}])$ is a smooth compact 1-dimensional manifold with boundary $\mathrm{ev}_{J_0}^{-1}(p) \sqcup \mathrm{ev}_{J_1}^{-1}(p)$, which implies that $[\mathrm{ev}_{J_0}^{-1}(p)] = [\mathrm{ev}_{J_1}^{-1}(p)]$ in $H_0(\overline{\mathcal{M}}_{0,1}(M, \omega_M, J_i; [\mathbb{P}^1 \times \{\mathrm{pt}\}]); \mathbb{Z})$.

This allows us to consider special ω_M-compatible almost complex structures on $M = \mathbb{P}^1 \times N$. For instance, we can choose a split almost complex structure $J = J_{\mathbb{P}^1} \times J_N$ for some chosen regular almost complex structure $J_N \in \mathcal{J}^{reg}_{\omega_N}(N)$. Using the projection $\pi_1\colon M \to \mathbb{P}^1$ and $\pi_2\colon M \to N$ and the fact that N is symplectically aspherical we can show that all non-constant J-holomorphic spheres in M must be contained in the fibres of π_2, representing the class $[\mathbb{P}^1] \times [\mathrm{pt}]$. The previous discussions for $\mathbb{P}^1$ then helps us to conclude that $[\mathrm{ev}_J^{-1}(p)] \neq 0$ in $H_0(\overline{\mathcal{M}}_{0,1}(M, \omega_M, J; [\mathbb{P}^1 \times \{\mathrm{pt}\}]); \mathbb{Z})$, which finishes the proof. □

Proof of theorem 3.3. Assume that there exists a symplectic embedding $i\colon D_r \hookrightarrow Z_R$, by shrinking the image of D_r if necessarily we can assume that the image is compact in Z_R, so that we can compactify Z_R into the product symplectic manifold $\mathbb{P}^1 \times \mathbb{T}^{2n-2}$ where the symplectic structure on $\mathbb{P}^1$ is of the form $(R+\varepsilon)\omega_{FS}$ for some $\varepsilon > 0$. As $\mathbb{T}^{2n-2}$ is symplectically aspherical by example 3.29, we can apply proposition 3.30 to conclude that there exists a J-holomorphic sphere $u\colon S \to M$ in $\mathbb{P}^1 \times \mathbb{T}^{2n-2}$ representing the class $[\mathbb{P}^1] \times [\mathrm{pt}]$ passing through any point of $\mathbb{P}^1 \times \mathbb{T}^{2n-2}$. This curve then clearly satisfy the identity

$$\int_S u^*\omega_M = \omega\left[\mathbb{P}^1 \times \{\mathrm{pt}\}\right] = (R+\varepsilon)^2\pi.$$

Let us pick $p = i(0) \in i(D_r)$ be the image of the origin, so that u has non-trivial intersection with $i(D_r)$. The pull-back $i^{-1}(u(S))$ still defines a i^*J-holomorphic curve in D_r passing through 0, and we know that if i^*J is linear, monotonicity lemma implies that

$$\pi r^2 \leq \int_{i^{-1}(u(S))} i^*\omega \leq \int_S u^*\omega_M = (R+\varepsilon)^2\pi,$$

which implies that $r \leq R$. We cannot prove that all such pull-backs are linear, but we can start with a linear almost complex structure J_0 on D_r, and consider an extension of i_*J_0 to $\mathbb{P}^1 \times \mathbb{T}^{2n-2}$, which still exists due to connectedness and contractibility of $\mathcal{J}_{\omega_M}(M)$ 2.5. The final step is to show that the space of almost complex structures with given behaviour on $i(D_r)$ is still big enough to give generic regularity, which can be achieved by keeping track of the proof of the regularity theorem, using the following two facts. This finishes the proof.

(i) The curve $u(S)$ cannot be completely contained in the image $i(D_r)$;

(ii) The proof of regularity requires the existence of almost complex structures with prescribed behaviour in a small neighbourhood of an injective point of u, which can be found in the complement of $i(D_r)$. □

2 PROBLEM. A symplectic 4-manifold is said to be **minimal** if it does not contain any symplectically embedded spheres with negative self-intersection number. Classify all compact minimal symplectic 4-manifolds containing a symplectically embedded sphere with non-negative self-intersection number.

Reference: [McD90]

3 PROBLEM. Consider the symplectic manifold $(\mathbb{P}^1 \times \mathbb{P}^1, a\omega_{FS} \oplus b\omega_{FS})$ where $a, b > 0$ are not necessarily equal. Compute $H^*(\mathrm{Symp}_0(\mathbb{P}^1 \times \mathbb{P}^1, a\omega_{FS} \oplus b\omega_{FS}); \mathbb{Z})$.

Reference: [Abr98]

4 PROBLEM. Compute the symplectic mapping class group of $T^*\mathbb{S}^2$.

Reference: [Sei98].

5 PROBLEM. This is a sequence of problems that lead to the proof of an almost hundred-year-long problem via symplectic geometry.

1. Suppose that (M, ω) is a symplectic manifold that is symplectomorphic to $\mathbb{R}^4$ outside a compact subset and is symplectically aspherical. Show that M is diffeomorphic to $\mathbb{R}^4$ ([Gro85]).
2. Show that there does not exist Lagrangian embedding of a Klein bottle into $\mathbb{R}^4$ ([Nem09]).
3. Let γ be a smooth Jordan curve in $\mathbb{R}^2$ and $R \subseteq \mathbb{R}^2$ a rectangle. Show that there exists a rectangle similar to R whose vertices lie on γ ([GL21]).

(3d) Gromov-Witten invariants. The idea behind Gromov's proof is to find existence of J-holomorphic spheres with given conditions via extracting invariants from the moduli space of J-holomorphic curves. This happened to have close relation with an emerging physics theory at that time, the well-known **string theory**. In short, a **quantum field theory** of dimension 2 has configuration space the space of smooth maps

$$C^\infty(\Sigma, M)$$

for some Riemann surface Σ into some smooth manifold M. Witten observed that when M is holomorphic, certain field theory of dimension 2 has $GL(n, \mathbb{C})$-symmetry, i.e. they are actually holomorphic field theories, so that the configuration space should be reduced to the space of holomorphic maps

$$\mathrm{Hol}((\Sigma, j), (M, J))$$

for some complex structure j on Σ and some complex structure J on M. Integrations on an infinite-dimensional manifolds are usually not defined, but $\mathrm{Hol}((\Sigma, j), (M, J))$ is usually finite dimensional, and therefore integration on $\mathrm{Hol}((\Sigma, j), (M, J))$ makes sense in a large scale. For given restrictions on the field theories, i.e. a bunch of cocycles

$\alpha_i \in H^i(M;\mathbb{Q})$ for $i = 1, 2, \cdots, k$, so that the particles have to interact, we define the **correlation function** of the field theory to be

$$\langle \alpha_1, \alpha_2, \cdots, \alpha_k \rangle_{g,A}^{GW} = \int_{[\mathcal{M}_{g,k}(M,\omega,J;A)]} \mathrm{ev}_1^* \alpha_1 \wedge \mathrm{ev}_2^* \alpha_2 \wedge \cdots \wedge \mathrm{ev}_k^* \alpha_k,$$

whenever the fundamental cycle $[\mathcal{M}_{g,k}(M,\omega,J;A)]$ is well-defined and the cocycles α_i has total degree equal to the expected dimension. These are the so-called **Gromov-Witten invariants** on M.

From the lecture we see that if (M,ω) is semipositive and $g = 0$, then the fundamental cycle $[\mathcal{M}_{g,k}(M,\omega,J;A)]$ is well-defined for regular ω-compatible almost complex structure J, and hence the Gromov-Witten invariants are well-defined. In general, we need to perturb the moduli space of J-holomorphic curves in order to get well-defined fundamental cycles, which we now call a **virtual fundamental cycle**. This was achieved in full generality in algebraic geometry [Beh97, BF97] for coefficients in $\mathbb{Q}$ both cohomologically and K-theoretically, and also in symplectic geometry [BX25, AMS24] recently for cohomologies with $\mathbb{Z}$- and some generalized cohomology coefficients.

When $k \geq 4$, the automorphism group of $\mathbb{P}^1$ no longer acts on $(\mathbb{P}^1)^{\times k}$ transitively, and hence the configuration space

$$\mathcal{M}_{0,k} = \left\{ (z_1, \cdots, z_k) \in \left(\mathbb{P}^1\right)^k \middle| z_i \neq z_j \text{ for } i \neq j \right\} / PGL_2(\mathbb{C})$$

is a smooth complex manifold of dimension $k-3$. We can naturally compactify $\mathcal{M}_{0,k}$ into a compact smooth manifold $\overline{\mathcal{M}}_{0,k}$, called **Grothendieck-Knudson manifold**, by adding stable curves of genus 0 as boundaries of $\mathcal{M}_{0,k}$.

3.33 Definition. An n-labelled tree is a tuple (T, E, Λ) where (T, E) is a tree and $\Lambda = \{\Lambda_\alpha\}_{\alpha \in T}$ is a decomposition of the index set $\{1, 2, \cdots, n\}$ into a disjoint union

$$\{1, 2, \cdots, n\} = \bigcup_{\alpha \in T} \Lambda_\alpha.$$

An n-labelled tree is called **stable** if for each α, $n_\alpha = \#\Lambda_\alpha + \#\{\beta \in T | (\alpha, \beta) \in E\} \geq 3$.

For each pair $(\alpha, \beta) \in E$ of vertices connected by an edge, let $z_{\alpha\beta} \in \mathbb{S}^2$ be a corresponding collection of points on the sphere $\mathbb{S}^2$, and let $z_1, \cdots, z_n \in \mathbb{S}^2$ be a collection of points with labelling $\alpha_i \in T$, then we call the tuple

$$\boldsymbol{z} = (\{z_{\alpha\beta}\}_{(\alpha,\beta) \in E}, \{\alpha_i, z_i\}_{1 \leq i \leq n})$$

a **stable curve** of genus 0 with n marked points if for each $\alpha \in T$ the points $z_{\alpha\beta}$ and z_i for $\alpha_i = \alpha$ are pairwise distinct.

15 Exercise. Try to determine the moduli space of 4 distinct points on $\mathbb{P}^1$, i.e. $\overline{\mathcal{M}}_{0,4}$.
Hard: What about spaces of 5 marked points?

The compactified moduli space $\overline{\mathcal{M}}_{0,k}(M,\omega,J;A)$ of stable J-holomorphic curves with k marked points in M naturally admits a forgetful map $\pi_0 \colon \overline{\mathcal{M}}_{0,k}(M,\omega,J;A) \to \overline{\mathcal{M}}_{0,k}$, and together with the evaluation maps we get a map to the product space

$$\overline{\mathcal{M}}_{0,k}(M,\omega,J;A) \to M^{\times k} \times \overline{\mathcal{M}_{0,k}}.$$

We define the Gromov-Witten invariants as the integration

$$GW^M_{0,k,A}(\alpha_1, \cdots, \alpha_k; \beta) = \int_{[\overline{\mathcal{M}}_{0,k}(M,\omega,J;A)]} \mathrm{ev}_1^* \alpha_1 \wedge \cdots \wedge \mathrm{ev}_k^* \alpha_k \wedge \pi_0^* \beta,$$

where $\beta \in H^*(\overline{\mathcal{M}}_{0,k}; \mathbb{Q})$ corresponds to “restrictions on the configurations of points” on $\mathbb{P}^1$.

Kontsevich-Manin [KM94] proposed a sequence of axioms for Gromov-Witten invariants which helps compute some of the Gromov-Witten invariants. Let’s simply list them here:

- (Effective) If $\omega(A) < 0$ then $GW^M_{0,k,A} = 0$.
- (Symmetry) For each permutation $\sigma \in \mathfrak{S}_k$ we have

  $$GW^M_{0,k,A}(\alpha_{\sigma(1)}, \cdots, \alpha_{\sigma(k)}; \beta) = \varepsilon(\sigma; \alpha) GW^M_{0,k,A}(\alpha_1, \cdots, \alpha_k; \sigma_* \beta)$$

  where

  $$\varepsilon(\sigma; \alpha) = (-1)^{\#\{i<j | \sigma(i) > \sigma(j),\ \deg \alpha_i \deg \alpha_j \in 2\mathbb{Z}+1\}}$$

  and $\sigma_* \beta$ permutes the orders of the marked points.
- (Grading) If $GW^M_{0,k,A}(\alpha_1, \cdots, \alpha_k; \beta) \neq 0$ then $\sum_{i=1}^k \deg \alpha_i - \deg \beta = 2n + 2c_1(A)$ where $2n = \dim M$.
- (Homology) For every $A \in H_*(M; \mathbb{Z})$ and every $k \geq 3$ there exists a homology class

  $$\sigma_{A,k} \in H_{2n+2c_1(A)+2k-6}(M^{\times k} \times \overline{\mathcal{M}}_{0,k})$$

  such that

  $$GW^M_{0,k,A}(\alpha_1, \cdots, \alpha_k; \beta) = \langle \mathrm{ev}_1^* \alpha_1 \cup \cdots \cup \mathrm{ev}_k^* \alpha_k \cup \pi_0^* \beta, \sigma_{A,k} \rangle.$$
- (Fundamental class)Let $\pi_{0,k} \colon \overline{\mathcal{M}}_{0,k}(M, \omega, J; A) \to \overline{\mathcal{M}}_{0,k}$ be the forgetful map, then

  $$GW^M_{0,k,A}(\alpha_1, \cdots, 1; \beta) = GW^M_{0,k-1,A}(\alpha_1, \cdots, \alpha_{k-1}; \pi_{0*} \beta).$$

  In particular, this invariant vanishes if $\beta = [\overline{\mathcal{M}}_{0,k}]$.
- (Divisor) If $(A, k) \neq (0, 3)$ and $\deg(\alpha_k) = 2$ then

  $$GW^M_{0,k,A}(\alpha_1, \cdots, \alpha_k; \beta) = \int_A \alpha_k \cdot GW^M_{0,k-1,A}(\alpha_1, \cdots, \alpha_{k-1}; \beta).$$
- (Zero) If $A = 0$ then $GW^M_{0,k}(\alpha_1, \cdots, \alpha_k; \beta) = 0$ whenever $\deg(\beta) > 0$ and

  $$GW^M_{0,k,0}(\alpha_1, \cdots, \alpha_k; [\mathrm{pt}]) = \int_M \alpha_1 \cup \cdots \cup \alpha_k.$$

- (Splitting) Fix a basis $e_0, \cdots, e_N$ of the cohomology $H^*(M)$, let

$$g_{\nu\mu} := \int_M e_\nu \cup e_\mu,$$

and denote by $g^{\nu\mu}$ the inverse matrix. Moreover, fix a partition of the index set $\{1, \cdots, k\} = S_0 \cup S_1$ such that $k_i := \#S_i \geq 2$ for $i = 0, 1$ and denote by

$$\phi_S \colon \overline{\mathcal{M}}_{0,k_0+1} \times \overline{\mathcal{M}}_{0,k_1+1} \to \overline{\mathcal{M}}_{0,k}$$

the canonical map that identifies the last marked point of a stable curve in $\overline{\mathcal{M}}_{0,k_0+1}$ with the first marked point of a stable curve in $\overline{\mathcal{M}}_{0,k_1+1}$, and the first k_0 marked points are sent to S_0 while the last k_1 are sent to S_1. Fix two homology classes $\beta_0 \in H_*(\overline{\mathcal{M}}_{0,k_0+1})$ and $\beta_1 \in H_*(\overline{\mathcal{M}}_{0,k_1+1})$, then

$$\begin{aligned} GW^M_{0,k,A}(\alpha_1, \cdots, \alpha_k; \phi_{S*}(\beta_0 \times \beta_1)) = \varepsilon(S, a) \sum_{A=A_0+A_1} \sum_{\nu,\mu} \\ GW^M_{0,k_0+1,A_0}(\alpha_{S_0}, e_\nu; \beta_0) g^{\nu\mu} GW^M_{0,k_1+1,A_1}(e_\mu, \alpha_{S_1}; \beta_1). \end{aligned}$$

6 Problem. Consider the complex manifold $\mathbb{P}^n$. We know that the only non-trivial spherical class in $\mathbb{P}^n$ is the embedded $\mathbb{P}^1 \hookrightarrow \mathbb{P}^n$. We denote it by L and write $N_d = GW^{\mathbb{P}^2}_{0,3d-1,dL}([\mathrm{pt}], \cdots, [\mathrm{pt}]; [\overline{\mathcal{M}}_{0,3d-1}])$ for the number of degree d rational curves in $\mathbb{P}^2$ passing through $3d-1$ generic points. Verify that the number N_d is determined by the initial condition $N_1 = 1$ and the recursion formula

$$N_d = \sum_{d_1+d_2=d} \left(d_1^2 d_2^2 N_{d_1} N_{d_2} \binom{3d-4}{3d_1-2} - d_1^3 d_2 N_{d_1} N_{d_2} \binom{3d-4}{3d_1-1} \right).$$

Reference: [MS12, Section 7.5].

7 Problem. A fibre bundle $F \to P \to B$ is **hamiltonian** if the structure group of P can be reduced to $\mathrm{Ham}(F)$, the group of Hamiltonian diffeomorphisms of F. Show that if $B = \mathbb{P}^n$, then there is an additive isomorphism of rational cohomology groups

$$H^*(P; \mathbb{Q}) \cong H^*(B; \mathbb{Q}) \otimes H^*(F; \mathbb{Q}).$$

Reference: [LM03].

When $k = 3$, there're no problems on moduli spaces of marked points, and we get a 3-point Gromov-Witten invariant

$$GW^M_{0,3,A} \colon H^*(M)^{\otimes 3} \to \mathbb{Z}.$$

When M is compact, using Poincaré duality we can identify $H^*(M)$ with $H_*(M)$, so that we obtain a bilinear map

$$H^*(M) \otimes H^*(M) \xrightarrow{GW_{3,A}} H^*(M).$$

This can be arranged into a ring structure on the singular cohomology

$$QH^*(M) \coloneqq H^*(M) \otimes_{\mathbb{Z}} \Lambda((q)),$$

where

$$\Lambda = \left\{ \sum_{i=0}^{\infty} \lambda_i T^{a_i} \,\middle|\, \# \left\{ \begin{array}{c} \lambda_i \neq 0 \\ a_i < c \end{array} \right\} < \infty \text{ for all } c > 0 \right\}$$

is the **Novikov ring** on M, and q is a formal variable of degree 2. The definition is given as follows: for $\alpha, \beta \in H^*(M)$ we define

$$\langle \alpha * \beta, \gamma \rangle = \sum_{A \in H_2(M)} GW^M_{0,3,A}(\alpha, \beta, \gamma) e^A q^{c_1(A)}.$$

This ring structure is called the **quantum cup product**, and the resulting ring is called the **quantum cohomology** of M.

8 PROBLEM. Compute the quantum cohomology of $\mathbb{P}^n$.
Reference: [MS12, Example 11.1.12].

4. Poincaré-Birkhoff, Arnold and Morse

In classical mechanics, one interesting problem is to understand motions of three planets moving in the universe where they're mutually affected by the gravitation. If one of the planets have mass much smaller than the other two, we can neglect the effect of the third planet and so the other two planets form the **two-body problem**, which is easy to solve and we have a complete understanding of it. This type of specification is called the **restricted three-body problem**.

Poincaré studied the restricted three-body problem and proposed a simple model that contains basic difficulties of this problem, which we now refer to as **Poincaré's Last Geometric Theorem**.

4.1 Theorem (Poincaré-Birkhoff). Let $A = \mathbb{S}^1 \times [0,1]$ be the closed annulus and $f\colon A \to A$ be an area-preserving diffeomorphism satisfying the twist condition, i.e. f preserves the orientation of one of the circles and reverses the other, then f must have at least two fixed points.

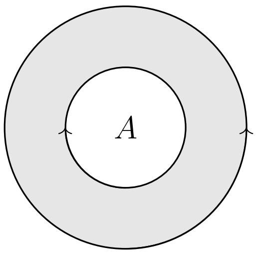

Figure 3: The Annulus and Twist Condition

This theorem was firstly proved by Poincaré in 1912 in some special cases, and Birkhoff carried out a complete proof in 1913 [Bir13]. We will not talk about their proof, but rather a method that connects this problem to the geometry of A, which can be generalized to any dimensions, due to Arnold [Ad65].

16 Exercise. Show that the twisting condition is essential, i.e. there exists an area-preserving diffeomorphism $f\colon A \to A$ without the twist condition such that f has no fixed point.

Hint: Rotations.

Note that $A \subseteq T^*\mathbb{S}^1$ is a symplectic submanifold with boundary, so there is a canonical 1-form $\lambda_A = \lambda_{\text{tau}}|_A = -pdq$ on A with $d\lambda_{\text{tau}} = dq \wedge dp$ defining a symplectic structure on A. We use this as our area form, so that any area-preserving diffeomorphisms $f\colon A \to A$ are in fact symplectomorphisms.

We use the cotangent bundle coordinates, so that $q \in \mathbb{S}^1$ is periodic, and for an example let's look at the smooth function $H\colon A \to \mathbb{R}$ defined by $H(q,p) = \dfrac{1}{2}p^2$.

17 Exercise. Show that the H defined here has a well-defined hamiltonian flow and satisfies the twist condition.

Note that $dH = 0$ iff $p = 0$, which implies that the critical points of H form the whole zero section $\mathbb{S}^1 \hookrightarrow A$, and in particular, the number of points on a circle is more than two points. Howver, we want to understand this behaviour for an arbitrary symplectomorphism $\varphi\colon A \to A$ with the twist condition.

(4a) Liouville manifolds and exact Lagrangians. Let's first try to understand the "twist condition" geometrically. In symplectic geometry this is related to the notion of "convexity" for boundaries of a symplectic manifold with boundary. Given a compact symplectic manifold (M, ω) with boundary ∂M. Write $\mathrm{Nbd}^M(\partial M) \cong M \times [-\varepsilon, 0] \hookrightarrow M$ be its collar neighbourhood.

4.2 Definition. ∂M is **convex** if there exists a vector field Z defined in $\mathrm{Nbd}^M(\partial M)$ such that $\mathcal{L}_Z \omega = \omega$ and Z is outward-pointing along ∂M.

One important class of convex symplectic manifolds is called Liouville domains, which we will define here:

4.3 Definition. Let (M, ω) be an exact compact symplectic manifold with boundary ∂M and let λ be a primitive of ω. We say that (M, λ) is a **Liouville domain** if the symplectic dual vector field Z of λ is outward-pointing along ∂M. The vector field Z is called the **Liouville vector field** of (M, λ), and the 1-form λ is called the **Liouville form** of (M, λ).

18 Exercise. Verify that the codisk bundle $(\mathbb{D}^*M, \lambda_{\mathrm{tau}})$ of a closed manifold M defined by

$$\{(x, v) \in T^*M \mid \|v\| \leq 1\},$$

where we choose a compatible almost complex structure J on T^*M and use the induced Riemannian metric to define the norm, is a Liouville domain with tautological 1-form as the Liouville 1-form.

19 Exercise. Show that the cotangent bundle T^*G of a Lie group G is trivial, i.e. there is a bundle isomorphism $T^*G \cong G \times \mathbb{R}^n$ where $n = \dim G$. Moreover, what is the symplectic form on $G \times \mathbb{R}^n$ under this isomorphism?

For instance, for $G = \mathbb{S}^1$, one can see that $\mathbb{D}^*\mathbb{S}^1 \cong A$ is symplectically the same as A, where the orientations of $\partial A = \mathbb{S}^1_{in} \sqcup \mathbb{S}^1_{out}$ are opposite to each other. Therefore a symplectomorphism $f\colon A \to A$ satisfying the twist condition is the same as an orientation-preserving symplectomorphism of the annulus A regarded as a Liouville domain.

20 Exercise. Show that $\mathrm{Diff}^+(A)$, the orientation-preserving diffeomorphism group of A, deformation restracts to the group $\mathrm{Symp}(A)$ of symplectomorphisms of A.

Proof. This is basically a Corollary of Moser's trick, where we need to find primitives of the derivative of the one-parameter family of symplectic forms directly. □

The topological facts about the structure of $\mathrm{Diff}^+(A)$ is not related to our lecture here, so let's just briefly summarize what conclusion we can obtain. Firstly, any diffeomorphism of A can be decomposed into two diffeomorphisms:

$$\mathrm{Diff}^+(A) = \mathrm{Diff}^+_c(A) \cdot \mathrm{Diff}^+_\infty(A),$$

where $\mathrm{Diff}_c^+(A)$ are diffeomorphisms of A which is identity near ∂A, and $\mathrm{Diff}_\infty^+(A)$ are orientation-preserving diffeomorphisms of A which is identity in a compact subset of $A \setminus \partial A$. Moreover, the **mapping class group** $\mathrm{Mod}(A) = \pi_0 \mathrm{Diff}_c^+(A)$ is known to be isomorphic to $\mathbb{Z}$, generated by the **Dehn twist** $\tau\colon A \to A$ defined by

$$\tau(r, \theta) = (r, \theta + \pi r)$$

[FM12, Proposition 2.4]. Since $\mathrm{Diff}^+(A)$ deformation retracts to $\mathrm{Symp}(A)$, and we can also show that $\mathrm{Diff}_c^+(A)$ also deformation retracts to $\mathrm{Symp}_c(A)$, the symplectomorphisms of A which is identity near ∂A with the same argument, we can further decompose $\mathrm{Symp}_c(A)$ into

$$\mathrm{Symp}_c(A) = \mathrm{Symp}_{c,0}(A) \cdot \mathrm{Mod}(A).$$

Therefore the proof of theorem 4.1 can be divided into anlysing fixed points of $f \in \mathrm{Symp}_{c,0}(A)$ and when f is a Dehn twist. For the first one, let's write down the statement as a separate proposition:

4.4 Proposition. For any symplectomorphism $f \in \mathrm{Symp}_{c,0}(A)$ with compact support and is symplectically isotopic to the identity, f has at least two fixed points in the interior support $\{x \in A \colon f_x \neq \mathrm{Id}\}$ of f.

Recall that there is a short exact sequence of vector spaces

$$0 \to \mathrm{ham}(A) \to \mathrm{symp}(A) \to H^1(A; \mathbb{R}) \to 0.$$

21 Exercise. Show that $\mathrm{Symp}_{c,0}(A) = \mathrm{Ham}_c(A)$, where $\mathrm{Ham}_c(A)$ is the group of Hamiltonian diffeomorphisms which are identity near ∂A.

Hint: What are the generators of $H^1_{dR}(A)$?

Note that $\varphi(\mathbb{S}^1) =: L$ are all Lagrangian submanifolds of A for any $\varphi \in \mathrm{Symp}_{c,0}(A)$. As $\omega_A|_L = 0$, the Liouville 1-form $\lambda_A|_L$ is a closed 1-form on L.

4.5 Definition. A Lagrangian submanifold $L \subseteq (M, \lambda)$ of a Liouville domain (M, λ) is called **exact** if $\lambda|_L$ is an exact 1-form on L, i.e. there exists a smooth function $f\colon L \to \mathbb{R}$ such that $\lambda|_L = df$.

A Lagrangian isotopy $i_t\colon L \to M$ is called **exact** if there exists a smooth function $f\colon [0,1] \times L \to \mathbb{R}$ such that $i_t^* \lambda = df_t$ for each $t \in [0,1]$, where $f_t(\cdot) = f(t, \cdot)$.

22 Exercise. Show that an exact Lagrangian isotopy is a Hamiltonian isotopy, i.e. there exists smooth functions $H_t\colon M \to \mathbb{R}$ such that the Hamiltonian flow $\varphi_{H_t}^t$ generated by H_t satisfies $\varphi_{H_t}^t(L) = i_t(L)$ for each $t \in [0,1]$.

As $A = T^*\mathbb{S}^1 \cong \mathbb{S}^1 \times \mathbb{R}$, it's very easy to find a smooth isotopy of $\mathbb{S}^1$ in A without any intersection with $\mathbb{S}^1$, and by isotopy extension this smooth isotopy extends to a compactly supported diffeomorphism of A, and hence we cannot deduce Proposition 4.4 in the smooth category. There has to be some essential symplectic information involved. This version of Poincaré-Birkhoff theorem was observed, generalized, conjectured and proved in a special case by Arnold in [Ad65], as we will state here:

4.6 Theorem (Arnold). Let $M = T^*Q$ be a cotangent bundle of a compact smooth manifold Q equipped with the standard Liouville 1-form $\lambda_M = \lambda_{\mathrm{tau}}$. Then for any exact Lagrangian submanifold $L \subseteq M$ that is a graph of Q, the intersection $L \cap Q$ has at least $\sum_i b_i(Q)$ points (counted with multiplicities).

In this section, we will explain Arnold's proof of the above theorem, via constructing explicitly homology theories for given generic smooth functions, called Morse homology. This is not the simplest way of proving this theorem, but will shed some lights on the subsequent lectures.

(4b) Morse functions. The main character in the proof is so-called Morse functions. For a smooth manifold M and a smooth function $f: M \to \mathbb{R}$, recall that a **critical point** of f is a point $p \in M$ such that $df_p = 0$. In this case, the restriction $TT^*M|_{df_p} \cong T_pM \oplus T_p^*M$ splits and so the second-order derivative $d^2 f_p$ of f at p is a well-defined matrix in $\mathrm{Mat}_{n\times n}(\mathbb{R})$ where $n = \dim M$. We say that p is **non-degenerate** if $d^2 f_p$ is non-degenerate as a matrix, i.e. $\det d^2 f_p \neq 0$.

4.7 Definition. Let M be a smooth manifold. A smooth function $f: M \to \mathbb{R}$ is called a **Morse function** if all its critical points are non-degenerate.

There're many Morse functions on a given smooth manifold, and in fact, the set of Morse functions is dense in the space of all smooth functions on M with respect to the C^∞-topology.

23 Exercise. Show that the set of Morse functions on M is dense in $C^\infty(M)$ with respect to the C^∞-topology.

Hint: Find an equivalent condition for a function to be Morse in terms of intersections of Lagrangian submanifolds in T^*M.

4.8 Lemma (Morse Lemma). Let M be a smooth manifold and $p \in M$ is a non-degenerate critical point of a smooth function $f: M \to \mathbb{R}$. Then there exists a local coordinate chart (U, φ) around p such that $\varphi(p) = 0$ and

$$f \circ \varphi^{-1}(x_1, \cdots, x_n) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2,$$

where λ is the number of negative eigenvalues of $d^2 f_p$ and is called the **Morse index** of f at p.

Proof. With a choice of a chart $\psi: U_0 \to \mathbb{R}^n$ with $\psi(p) = 0$, the function $f \circ \psi^{-1}(x_1, \cdots, x_n)$ is a smooth function on $\psi(U_0)$ with a Taylor expansion near 0 whose first three orders can be written as

$$f \circ \psi^{-1}(x_1, \cdots, x_n) = f(p) + \sum_{i,j} a_{ij} x_i x_j + O(\|x\|^3),$$

and the task is to find a local diffeomorphism $\Psi: \mathrm{Nbd}_0 \to \mathrm{Nbd}_0$ to cancel out all the correction terms and turn the Hessian matrix into the standard normal form.

We adapt here a pretty algebraic but clear approach to construct such diffeomorphisms. Define

$$C^\infty(\mathbb{R}^n)_0 = \frac{\{(U, f) | 0 \in U \text{ open}, \ f \in C^\infty(U)\}}{(U, f) \sim (V, g) \text{ iff } f|_{U\cap V} = g|_{U\cap V}},$$

then we can verify that $C^\infty(\mathbb{R}^n)_0$ is a commutative ring where multiplication is multiplying the functions and restricting to their intersections. This ring has a maximal ideal of the form

$$\mathfrak{m}_0 = \{(U, f) \in C_0^\infty(U) | f(0) = 0\}.$$

One can verify that for any $k \geq 0$, we can identify

$$C^\infty(\mathbb{R}^n)_0/\mathfrak{m}_0^{k+1} \cong \left\{ a_0 + \sum_{i=1}^n a_i x_i + \cdots + \sum_{i_1+\cdots+i_n=k} a_{i_1\cdots i_n} \prod_{j=1}^n x_j^{i_j} \right\}$$

the quotient of $C^\infty(\mathbb{R}^n)_0$ by $\mathfrak{m}_0^k$ with the space of Taylor expansions of any smooth function $f \in C^\infty(\mathbb{R}^n)_0$ near 0 up to order k.

Write

$$Q(x_1, \cdots, x_n) = f(p) + \sum \frac{\partial^2 f \circ \psi^{-1}}{\partial x_i \partial x_j} x_i x_j$$

be the Hessian of f at p. Set $f_t = (1-t) f \circ \psi^{-1} + tQ$. We want to look at a smooth family of diffeomorphisms $\Phi^t \colon U \to \Phi^t(U)$ in open subsets of $\mathbb{R}^n$ satisfying $\Phi^t(0) = 0$ and $\Phi^0 = \mathrm{id}$ such that $f_t \circ \Phi^t = f_0$. Note that any such diffeomorphism Φ^t induces a ring map $C^\infty(\mathbb{R}^n)_0 \to C^\infty(\mathbb{R}^n)_0$ and hence preserves the maximal ideal $\mathfrak{m}_0$. Set $V_t = \partial_t \Phi^t$ to be the derivative of Φ^t with respect to t, which is a smooth family of vector fields on $\mathbb{R}^n$ vanishing at 0, which also defines $C_0^\infty(M)$-module homomorphisms $\mathfrak{m}_0^{k+1} \to \mathfrak{m}_0^k$ for each $k \geq 0$. As $f_t \circ \Phi^t \equiv f_0$, we have

$$V_t(f_t) + \partial_t f_t \circ \Phi^t = 0,$$

and therefore we aim to solve the equation $V_t(f_t) = -\partial_t f_t \circ \Phi^t$ for V_t. Note that $\partial_t f_t = Q - f \circ \psi^{-1} := \delta$ has zeroes of order 3 at 0, so that $\partial_t f_t \circ \Phi^t \in \mathfrak{m}_0^3$ as an element in $C^\infty(\mathbb{R}^n)_0$. On the other hand, as V_t maps constant functions to 0, we have $V_t(f_t) = V_t(f_0 + t\delta) \in \mathfrak{m}_0^1$. We are therefore looking at the initial value problem

$$\begin{cases} (V_t)_0 = 0 \\ V_t(f_0 + t\delta) = g \end{cases} \tag{4}$$

for any given $g \in C^\infty(\mathbb{R}^n)_0$. As $V_t(f_0 + t\delta) \in \mathfrak{m}_0$, we must have $g \in \mathfrak{m}_0$. As $g(0) = 0$, we can write

$$g(x) = \int_0^1 \frac{d}{ds} g(sx) ds = \sum_{i=1}^n x_i \int_0^1 \frac{\partial g}{\partial x_i}(sx) ds = \sum_{i=1}^n x_i g_i(x)$$

for some $g_i \in C^\infty(\mathbb{R}^n)_0$, and hence it suffices to show existence of solutions for $g = x_i$, $1 \leq i \leq n$. In order to simplify the computation, we want to describe $\mathfrak{m}_0$ in terms of partial derivatives of f_0. Write J_{f_0} be the ideal generated by partial derivatives of f_0, i.e. $\dfrac{\partial f_0}{\partial x_i}$ for each $1 \leq i \leq n$. The non-degeneracy of $d^2 f_0$ implies that

4.9 Lemma. $J_{f_0} = \mathfrak{m}_0$, and hence partial derivatives of f_0 also forms a set of generators of $\mathfrak{m}_0$.

The change of basis matrix is denoted by $(a_{ij}(x))$ where

$$x_i = \sum_{j=1}^n a_{ij} \frac{\partial f_0}{\partial x_j},$$

which can be written in matrix form as $\vec{x} = A(x)\nabla f_0 = A(x)\nabla(f_0 - t\delta) + tA(x)\nabla\delta$. As $\delta \in \mathfrak{m}_0^3$, we can find a matrix $B \in \mathrm{Mat}_{n\times n}(\mathfrak{m}_0)$ such that $\nabla\delta = B(x)\vec{x}$, and hence we have the equation

$$(\mathrm{Id} - tA(x)B(x))\vec{x} = A(x)\nabla(f_0 - t\delta).$$

Note that $tA(x)B(x) \in \mathrm{Mat}_{n\times n}(\mathfrak{m}_0)$, up to taking a smaller neighbourhood we conclude that $\mathrm{Id} - tA(x)B(x)$ is invertible, hence we have

$$\vec{x} = (\mathrm{Id} - tA(x)B(x))^{-1}A(x)\nabla(f_0 - t\delta).$$

This proves the existence of solutions for $g \in \mathfrak{m}_0$ without the requirement that $V_t(0) = 0$. To get this requirement, we need $g \in \mathfrak{m}_0^2$, which is also true for δ.

Finally, by applying a linear change of basis we can turn the Hessian matrix Q of f_0 into the standard normal form, which finishes the proof. □

This approach of proof is called **homotopy method**, which is also useful in proving a lot of different but similar flexibility results. Interesting readers can consult [AdGiZV85] for more information.

Morse lemma implies that critical points of a smooth manifold M are isolated, and hence if M compact, there're only finitely many critical points.

4.10 Definition. Let $f\colon M \to \mathbb{R}$ be a smooth function on a smooth manifold M. A vector field $V \in C^\infty(M, TM)$ is called a **gradient-like vector field** for f if $V(f) \geq 0$ everywhere on M and $V(f) = 0$ iff $df = 0$. Conversely, for a given vector field V on M, a smooth function $f\colon M \to \mathbb{R}$ with the above property is called **Lyapunov** for V.

For each critical point p and the given gradient-like vector field V, the **stable manifold** $W^s(p)$ of p is the set of points $x \in M$ such that

$$\lim_{t\to+\infty} \phi_V^t(x) = p,$$

and the **unstable manifold** $W^u(p)$ of p is the set of points $x \in M$ such that

$$\lim_{t\to-\infty} \phi_V^t(x) = p.$$

24 Exercise. Show that for a given critical point p, the stable manifold $W^s(p)$ is an embedded (not closed) submanifold of M with dimension $\mathrm{Ind}(f)$, the Morse index of f, and the unstable manifold $W^u(p)$ is an embedded submanifold of M with dimension $\dim M - \mathrm{Ind}(f)$.

Let p and q be two critical points of f, then it's clear that $W^u(p) \cap W^s(q)$ is the set of points $x \in M$ lying in trajectories of the gradient-like vector field V going from p to q. Note that it starts from a positive eigenvalue of the Hessian of f at p and ends at a negative eigenvalue of the Hessian of f at q, so we expect that in general the Morse index of critical points decreases along trajectories of V. This is not true in general, as depicted in the following example:

4.11 Example. Consider the two-dimensional torus $\mathbb{T}^2$ with a height function $h\colon \mathbb{T}^2 \to \mathbb{R}$ defined by firstly embedding $\mathbb{T}^2$ to $\mathbb{R}^3$ via the map

$$(\theta, \vartheta) \mapsto ((2 + \cos\vartheta)\cos\theta, \sin\vartheta, (2 + \cos\vartheta)\sin\theta),$$

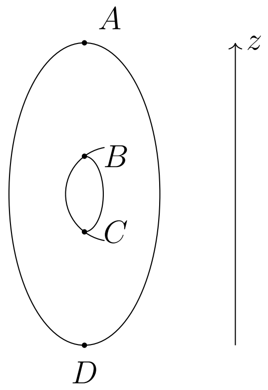

Figure 4: Height function on a torus.

and consider the height function $h(x, y, z) = z$. The critical points of h are depicted in figure 4.

Gradient-like vector fields on $\mathbb{T}^2$ with respect to h are clearly projections of the vector field $\partial/\partial z$ on $\mathbb{R}^3$ to $\mathbb{T}^2$ via the standard Riemannian metric, and we see that there're two flow lines going from point C to point B, but on the other hand we can compute that B and C have Morse index 1.

25 Exercise. Compute the Morse index of all critical points in the above example.

However, we are able to perturb the Morse function or the gradient-like vector field to make stable and unstable manifolds intersect transversely.

4.12 Theorem. A Morse pair (f, V) of a Morse function $f\colon M \to \mathbb{R}$ and a gradient-like vector field V is called **Morse-Smale** if for any two critical points p and q, the stable manifold $W^s(p)$ and the unstable manifold $W^u(q)$ intersect transversely. For any Morse function $f\colon M \to \mathbb{R}$ on a compact manifold M, there exists a Morse-Smale pair (f', V') such that f' is arbitrarily close to f in the C^∞-topology and V' is a gradient-like vector field for f'.

Proof. Given a pair of critical points p and q, at a point $x \in W^u(p) \cap W^s(q)$ the stable and unstable manifolds intersect transversely iff they intersect transversely at all points $\phi_V^t(x)$ for $t \in \mathbb{R}$, and hence after perturbing f so that critical points of f lie in different levels, it suffices to perturb f and V in regular level sets of f between critical points.

The intersections $W^u(p) \cap f^{-1}(r)$ and $W^s(q) \cap f^{-1}(r)$ are embedded spheres of dimensions $\mathrm{Ind}(p) - 1$ and $\dim M - \mathrm{Ind}(q) - 1$ respectively, and the stable (resp. unstable) manifolds of different critical points are disjoint from each other, so transversality theorem implies that after isotoping the stable spheres we can achieve that all stable spheres are transversal to all unstable spheres, and via isotopy extension theorem this can be extended to a smooth isotopy of the regular level set $f^{-1}(r)$.

Write this isotopy as $h_t\colon f^{-1}(r) \times [0, 1] \to f^{-1}(r)$ with $h_0 = \mathrm{Id}$ and h_1 makes all stable spheres transversal to all unstable spheres, we can extend it to a global diffeomorphism $H\colon M \to M$ which is h_1 at $f^{-1}(r)$ and is identity outside $f^{-1}(r - \varepsilon, r + \varepsilon)$ for some small $\varepsilon > 0$. This global diffeomorphism then pushes the gradient vector field V to a new vector field $V' = H_* V$ which is gradient-like for the new Morse function $f' = f \circ H^{-1}$, where the stable and unstable spheres at $f^{-1}(r)$ are transversal. As there're discretely many critical points, we only need to perturb finitely many times. □

From now on, we will always assume a Morse function $f\colon M\to\mathbb{R}$ is Morse-Smale and fix a corresponding gradient-like vector field V.

(4c) Morse homology. For a given Morse-Smale pair (f,V) on a smooth manifold M, we can define a chain complex $C_*(f;\Bbbk)$, where $\Bbbk$ is a ring, as follows. For each $k\in\mathbb{Z}$, the k-th chain group $C_k(f;\Bbbk)$ is the free $\Bbbk$-module generated by critical points of f with Morse index k. To define the boundary map, for each pair of critical points p and q with $\mathrm{Ind}(p)>\mathrm{Ind}(q)$, we consider the space

$$\widehat{\mathcal{M}}(p,q):=\left\{x\in M\,\middle|\,\lim_{t\to-\infty}\phi_V^t(x)=q\text{ and }\lim_{t\to+\infty}\phi_V^t(x)=p\right\}=W^u(q)\cap W^s(p).$$

Morse-Smale property implies that $\widehat{\mathcal{M}}(p,q)$ is a smooth submanifold of M of dimension $\mathrm{Ind}(p)-\mathrm{Ind}(q)$, where the flow of V provides a free $\mathbb{R}$-action on $\widehat{\mathcal{M}}(p,q)$. Taking quotient by $\mathbb{R}$-action provides us a manifold $\mathcal{M}(p,q)$ of dimension $\mathrm{Ind}(p)-\mathrm{Ind}(q)-1$.

4.13 Proposition (Compactness). Suppose $f\colon M\to\mathbb{R}$ is proper, then for any sequence of points $\{x_n\}\subseteq\mathcal{M}(p,q)$, there exists $k\geq 0$ and critical points $c_1,\cdots,c_k$ with $\mathrm{Ind}(p)>\mathrm{Ind}(c_1)>\cdots>\mathrm{Ind}(c_k)>\mathrm{Ind}(q)$ and a subsequence $\{x_{n_k}\}$ such that x_{n_k} converges to a broken trajectory from p to q through $c_1,\cdots,c_k$, i.e. there exists $y_1\in\mathcal{M}(p,c_1)$, $y_i\in\mathcal{M}(c_{i-1},c_i)$ for $2\leq i\leq k$ and $y_{k+1}\in\mathcal{M}(c_k,q)$ such that $\lim_{k\to\infty}x_{n_k}=y_1\#y_2\#\cdots\#y_{k+1}$, where $y_1\#y_2\#\cdots\#y_{k+1}$ is the concatenation of the trajectories represented by $y_1,\cdots,y_{k+1}$.

Proof. Since f is proper, $\widehat{\mathcal{M}}(p,q)\subseteq f^{-1}[f(q),f(p)]$ is precompact, and hence any integral curve of V must either converging to some point in $f^{-1}[f(q),f(p)]$ or touching the boundary of $f^{-1}([f(q),f(p)])$.

Let $\mathrm{Nbd}(p)$ be a Morse chart for p, then $x_n(t)$ intersects with $\partial\,\mathrm{Nbd}(p)$ at a point $x_n(t_{n,-})$ in the intersection $W^u(p)\cap\partial\,\mathrm{Nbd}(p)$, which is diffeomorphic to a sphere of dimension $\mathrm{Ind}(p)-1$, and is hence compact. Similarly $x_n(t)$ intersects with $\partial\,\mathrm{Nbd}(q)$ at some point $x_n(t_{n,+})$ in the intersection $W^s(q)\cap\partial\,\mathrm{Nbd}(q)$, so that up to passing to subsequences we can assume that $\lim_n x_n(t_{n,-})=y_-$ and $\lim_n x_n(t_{n,+})=y_+$ for some points y_-, y_+ on the corresponding spheres.

Let $y_1(t)$ be the integral curve of V passing through y_-, then properness of f implies that either there exists t_0 so that $f(y_1(t_0))=f(q)$ or $\lim_{t\to\infty}y(t)=c_1$ is some critical point c_1 of f. To rule out the first case, we use the following lemma:

4.14 Lemma. Suppose that $x\in M\setminus\mathrm{Crit}(f)$ and $\{x_n\}$ be a sequence of points tending to x. Let $\{y_n\}$ and y be points lying on the same trajectory of V as x_n and x, respectively. If $f(y)=f(y_n)$, then $\lim\limits_{n\to\infty}y_n=y$.

In particular, there must exist $t_1<t_0$ so that $f(y_1(t_1))=f(x_n(t_{n,+}))$, so that by Lemma 4.14 $x_n(t_{n,+})\to y_1(t_1)$ and hence $y_1(t_1)=y_+$ lies on the stable sphere of q, and therefore $\lim_{t\to\infty}y_1(t)=q$.

Let $\mathrm{Nbd}(c_1)$ be a Morse chart near c_1, with $y_1(t)$ intersecting with the stable sphere of c_1 at point $y_1(t_{1,+})$, then by Lemma 4.14 we must have $x_n(t_{n,1,+})\to y_1(t_{1,+})$ where $x_n(t_{n,1,+})$ is the point lying on the level set $f^{-1}(f(y_1(t_{1,+})))$. Therefore x_n enters the Morse chart of c_1 for n sufficiently large, and hence must exit $\mathrm{Nbd}(c_1)$ at some other time $t_{n,2,-}$. As $\partial\,\mathrm{Nbd}(c_1)$ is compact, up to further passing to subsequences we can assume that $x_n(t_{n,2,-})\to y_2(t_{2,-})$ for some point $y_2(t_{2,-})$ on $\partial\,\mathrm{Nbd}(c_2)$. Let $y_2(t)$ be the integral curve of V passing through $y_2(t_{2,-})$. Lemma 4.14 implies that $\lim_{t\to-\infty} y_2(t) = c_1$, and for similar reasons $\lim_{t\to\infty} y_2(t) = c_2$ for some other critical points c_2 of f.

Properness of f and discreteness of critical points imply that this procedure ends at some finite steps, providing integral curves $y_1, y_2, \cdots, y_{k+1}$ connecting critical points $p, c_1, c_2, \cdots, c_k, q$ so that a subsequence of $\{x_n\}$ converges to the concatenation $y_1\#y_2\#\cdots\#y_{k+1}$. □

Proof of Lemma 4.14. Set $U = M\setminus\mathrm{Crit}(f)$ be the open subset away from critical points, and consider the normalized gradient-like vector field $X = -\dfrac{1}{df(V)}V$, then X is well-defined in U whose flow satisfies

$$f\circ\psi_X^t(z) = f(z) - t,\ \forall z\in U.$$

We then know that

$$y_n = \psi_X^{f(y_n)-f(x_n)}(x_n) = \psi_X^{f(y)-f(x_n)}(x_n)$$

and

$$y = \psi_X^{f(y)-f(x)}(x).$$

Continuity of ψ_X^t and f then implies that $\lim_n y_n = y$. □

Compactness result implies that the space of broken trajectories

$$\mathcal{M}(p,c_1)\times\mathcal{M}(c_1,c_2)\times\cdots\times\mathcal{M}(c_k,q)$$

provides natural possible C^0-limits of $\mathcal{M}(p,q)$, but we still need to investigate whether all broken trajectories can be obtained as limits of trajectories in $\mathcal{M}(p,q)$. This is what we usually called **gluing analysis**, which in this case can be proved directly.

4.15 Proposition (Gluing). Let

$$\overline{\mathcal{M}}(p,q) = \bigsqcup_{\substack{k\geq 0\\ \mathrm{Ind}(p)>\mathrm{Ind}(c_1)>\cdots>\mathrm{Ind}(c_k)>\mathrm{Ind}(q)}} \mathcal{M}(p,c_1)\times\mathcal{M}(c_1,c_2)\times\cdots\times\mathcal{M}(c_k,q),$$

regarded as quotient of a subspace of M by the $\mathbb{R}$-action on each component, then $\overline{\mathcal{M}}(p,q)$ is a smooth manifold with corners, and the interior of $\overline{\mathcal{M}}(p,q)$ is $\mathcal{M}(p,q)$.

Proof. Note that the product

$$\mathcal{M}(p,c_1)\times\mathcal{M}(c_1,c_2)\times\cdots\times\mathcal{M}(c_k,q)$$

is a smooth manifold of dimension $\mathrm{Ind}(p)-\mathrm{Ind}(q)-k-1$, and hence we want to regard it as the codimension k corner stratum of $\overline{\mathcal{M}}(p,q)$. Let's start with looking at a baby example, which will quickly imply the whole theorem:

4.16 Lemma. Suppose that p,c,q are critical points of f with $\mathrm{Ind}(p)>\mathrm{Ind}(c)>\mathrm{Ind}(q)$, then there exists a smooth embedding $\chi\colon [0,\varepsilon)\times\overline{\mathcal{M}}(p,c)\times\overline{\mathcal{M}}(c,q)\to\overline{\mathcal{M}}(p,q)$ such that $(0,\varepsilon)\times\overline{\mathcal{M}}(p,c)\times\overline{\mathcal{M}}(c,q)$ is mapped to $\mathcal{M}(p,q)$ and $\{0\}\times\overline{\mathcal{M}}(p,c)\times\overline{\mathcal{M}}(c,q)$ is identified with $\overline{\mathcal{M}}(p,c)\times\overline{\mathcal{M}}(c,q)$.

Let $\mathrm{Nbd}(c)$ be a Morse chart for c. We write $\partial_+ \mathrm{Nbd}(c)$ for the outward boundary of c and $\partial_- \mathrm{Nbd}(c)$ for the inward boundary of c. Let $S^+(c) = W^u(c) \cap \partial_+ \mathrm{Nbd}(c)$ and $S^-(c) = W^s(c) \cap \partial_- \mathrm{Nbd}(c)$ be the stable and unstable spheres of c respectively, which are spheres of dimensions $\mathrm{Ind}(c) - 1$ and $\dim M - \mathrm{Ind}(c) - 1$ respectively.

Transversality of $W^s(p)$ with $W^u(c)$ provides a smooth submanifold $\mathcal{P}(c,p) = W^s(p) \cap S^+(c)$ of $W^s(p)$ of dimension $\mathrm{Ind}(p) - \mathrm{Ind}(c) - 1$, and hence its codimension as a submanifold of $W^s(p)$ is $\mathrm{Ind}(c)$. Pick a Riemannian metric on M and consider a tubular neighbourhood of $\mathcal{P}(c,p)$ in $W^s(p)$. Then for a chosen trajectory $x_1 \in \mathcal{M}(p,c)$, we can find an open ball D_{x_1} of small radius in $W^s(p) \cap S^+(c)$ containing the point $p_1 = x_1(\mathbb{R}) \cap S^+(c)$, which clearly has dimension $\mathrm{Ind}(c)$. Now we flow the punctured disk $D_{x_1} \setminus p_1$ to the negative boundary $\partial_- \mathrm{Nbd}(c)$ and denote the resulting subset by $\Phi(D_{x_1} \setminus p_1)$. We claim that

4.17 Lemma. *$S_-(c)$ is the boundary of $\Phi(D_{x_1} \setminus p_1)$ in $\partial_- \mathrm{Nbd}(c)$, making the union $\Phi(D_{x_1} \setminus p_1) \cup S^-(c)$ into a smooth manifold with boundary.*

Once this is established, the unstable manifold $W^u(q)$ intersects transversely with $S^-(c)$ along a submanifold of dimension $\mathrm{Ind}(c) - \mathrm{Ind}(q) - 1$, and with $\Phi(D_{x_1} \setminus p_1)$ along a submanifold of dimension $\mathrm{Ind}(c) - \mathrm{Ind}(q)$. Note that $S^-(c) \cap W^u(q)$ is in bijection with the set of all trajectories $x_2 \in \mathcal{M}(q,c)$, and hence collar neighbourood of $S^-(c) \cap W^u(q)$ in $W^u(q) \cap (\Phi(D_{x_1} \cap p_1) \cup S^-(c))$ provides the desired gluing map. □

Proof of Lemma 4.17. To proceed the proof, it suffices for us to look at the local Morse chart $\mathrm{Nbd}(c) \hookrightarrow \mathbb{R}^n$ of c. Locally c can be identified with 0 where the Morse function f is of the form

$$f(x) = f(c) + \|x_+\|^2 - \|x_-\|^2,$$

and the gradient-like vector field V can be chosen to be

$$V = \nabla f = 2\vec{x}_+ \cdot \frac{\partial}{\partial \vec{x}_+} - 2\vec{x}_- \cdot \frac{\partial}{\partial \vec{x}_-}.$$

The stable/unstable spheres $S^+(c)$ and $S^-(c)$ in this chart can be written as

$$S^+(c) = \{(\vec{x}_+, 0) | \|\vec{x}_+\| = \varepsilon\} \subseteq f^{-1}(f(c)+\varepsilon), \quad S^-(c) = \{(0, \vec{x}_-) | \|\vec{x}_-\| = \varepsilon\} \subseteq f^{-1}(f(c)-\varepsilon).$$

The disk D_{x_1} is a disk of dimension $\mathrm{Ind}(c)$ in $\partial_+ \mathrm{Nbd}(c) = f^{-1}(f(c) + \varepsilon)$ that intersects transversely with $S^+(c)$ at the point p_1, so that we can write D_{x_1} as

$$D_{x_1} = \{(h(x_-), x_-) | x_- \in D_0(\delta)\}$$

for some $\delta > 0$ and some smooth function $h \colon \mathbb{R}^{\mathrm{Ind}(c)} \to \mathbb{R}^{\dim M - \mathrm{Ind}(c)}$, up to shrinking D_{x_1} if possible. The flow of V acting on points in $D_{x_1} \setminus p_1$ can be written as

$$\phi_V^t(x) = \left(e^{2t} h(x_-), e^{-2t} x_-\right).$$

If $x \in \partial_+ \mathrm{Nbd}(c)$ with $x_- \neq 0$, then we can find t such that

$$\left(\frac{\|x_-\|}{\|h(x_-)\|} h(x_-), \frac{\|h(x_-)\|}{\|x_-\|} x_-\right) = \left(e^{2t} h(x_-), e^{-2t} x_-\right).$$

As $\|x_+\| > \|x_-\|$ in this disk, we have $t < 0$ and $\left(\frac{\|x_-\|}{\|h(x_-)\|} h(x_-), \frac{\|h(x_-)\|}{\|x_-\|} x_-\right) \in \partial_- \mathrm{Nbd}(c)$. Note that as $f(h(x_-), x_-) = f(c) + \varepsilon$, we have

$$\left(\frac{\|x_-\|}{\|h(x_-)\|} h(x_-), \frac{\|h(x_-)\|}{\|x_-\|} x_-\right) = \left(\frac{\|x_-\|}{\sqrt{f(c) + \varepsilon - \|x_-\|^2}} h(x_-), \frac{\sqrt{f(c) + \varepsilon - \|x_-\|^2}}{\|x_-\|} x_-\right).$$

If we use polar coordinates $(\rho, \theta) \in (0, \delta) \times \mathbb{S}^{\mathrm{Ind}(c)-1}$ for x_-, then the above map can be written as

$$(\rho, \theta) \mapsto \left(\frac{\rho}{\sqrt{f(c) + \varepsilon - \rho^2}} h(\rho\theta), \sqrt{f(c) + \varepsilon - \rho^2} \cdot \theta \right).$$

This map naturally extends to $\rho = 0$, with image in coordinates as

$$\left\{ \left(0, \sqrt{f(c) + \varepsilon} \cdot \theta\right) \middle| \theta \in \mathbb{S}^{\mathrm{Ind}(c)-1} \right\} = S^-(c).$$

This justifies the claim. $\square$

The last ingredient of the construction is about orientation. Recall that for a point $p \in M$, we can define an **orientation** on the tangent space T_pM as a choice of an element in the quotient space $\det(T_pM) \setminus \{0\}/\mathbb{R}_+$, where $\mathbb{R}_+$ acts by scaling. We denote the choice of orientation at p by o_p. For a smoot manifold M, an **orientation** on M is a choice of sections over the quotient bundle $(\det(TM) \setminus M)/\mathbb{R}_+$. As $\widehat{\mathcal{M}}(p, q) = W^s(p) \cap W^u(q)$, we have the short exact sequence of bundles

$$0 \to T\widehat{\mathcal{M}}(p, q) \to TW^s(p)|_{\widehat{\mathcal{M}}(p,q)} \oplus TW^u(q)|_{\widehat{\mathcal{M}}(p,q)} \to TM|_{\widehat{\mathcal{M}}(p,q)} \to 0,$$

which after taking det implies that

$$\det TW^s(p)|_{\widehat{\mathcal{M}}(p,q)} \otimes \det TW^u(q)|_{\widehat{\mathcal{M}}(p,q)} \cong \det TM|_{\widehat{\mathcal{M}}(p,q)} \otimes \det T\widehat{\mathcal{M}}(p, q).$$

On the other hand, at critical points p, q we have decompositions

$$\det T_pM \cong \det T_pW^s(p) \otimes \det T_pW^u(p), \quad \det T_qM \cong \det T_qW^s(q) \otimes \det T_qW^u(q).$$

Therefore we get an identification

$$\det T\widehat{\mathcal{M}}(p, q) \cong \det TW^s(p)|_{\widehat{\mathcal{M}}(p,q)} \otimes \det TW^s(q)|^{-1}_{\widehat{\mathcal{M}}(p,q)}.$$

Note that $W^s(p)$ are smooth disks, and hence a choice of orientation on $TW^s(p)$ is the same as a choice of orientation on $T_pW^s(p)$. Therefore a choice of orientation on $\det T\hat{\mathcal{M}}(p, q)$ is determined by choices of orientations of $TW^s(p)$ at each critical point p. This allows us to provide orientations to each $\mathcal{M}(p, q)$ and to spaces of broken trajectories as well, together with **coherence** between different strata of $\overline{\mathcal{M}}(p, q)$, i.e. boundaries of $\overline{\mathcal{M}}(p, q)$ must inherit boundary orientation from its interior $\mathcal{M}(p, q)$.

A trajectory $x \in \mathcal{M}(p, q)$ is the same as a map $x \colon \mathbb{R} \to \widehat{\mathcal{M}}(p, q)$, so that we get an isomorphism of bundles

$$x^* \det TW^s(q) \otimes x^* \det T\widehat{\mathcal{M}}(p, q) \cong x^* \det TW^s(p).$$

Therefore an orientation on x provides a map $x^* \det TW^s(q)/\mathbb{R}_{>0} \to x^* \det TW^s(p)/\mathbb{R}_{>0}$, and comparing with the orientations chosen on p and q, we obtain a sign $\pm$ for x which indicates that it preserves or reverses the orientation chosen on p.

We are now ready to define the boundary map $\partial \colon C_k(f; \Bbbk) \to C_{k-1}(f; \Bbbk)$. For $p \in \mathrm{Crit}(f)$ with Morse index k, we define

$$\partial(p) = \sum_{\substack{q \in \mathrm{Crit}(f) \\ \mathrm{Ind}(q) = k-1}} (\#\mathcal{M}(p, q))\, q,$$

where $\mathcal{M}(p,q)$ is a compact zero-dimensional manifold where each element $x \in \mathcal{M}(p,q)$ is equipped with a sign $\pm$ as above, and hence the signed count $\#\mathcal{M}(p,q) = \chi\left(H_*(\mathcal{M}(p,q)); \Bbbk\right)$ makes sense. To check that it defines a complex, note that $\partial^2(p)$ is given by the sum

$$\partial^2(p) = \sum_{\substack{r \in \mathrm{Crit}(f) \\ \mathrm{Ind}(r)=k-2}} \left(\sum_{\substack{q \in \mathrm{Crit}(f) \\ \mathrm{Ind}(q)=k-1}} \#\mathcal{M}(p,q) \cdot \#\mathcal{M}(q,r) \right) r,$$

where the sum of the counts $\sum \#\mathcal{M}(p,q)\cdot\#\mathcal{M}(q,r)$ is the same as the count of the broken trajectories in $\overline{\mathcal{M}}(p,r) \setminus \mathcal{M}(p,r)$, which is the boundary of the compact one-dimensional manifold $\overline{\mathcal{M}}(p,r)$, and hence the count is zero.

4.18 Definition. We define the **Morse homology** of a Morse-Smale pair (f,V) to be the homology of the chain complex $(C_*(f;\Bbbk), \partial)$, and denote it by $HM_*(M,f;\Bbbk)$.

(4d) Properties of Morse homology. After the construction of Morse homology, we first want to verify that

4.19 Proposition. Let (f,V) and (g,W) be two Morse-Smale pairs on a smooth manifold M, then there exists a quasi-isomorphism of chain complexes

$$C_*(f;\Bbbk) \cong C_*(g;\Bbbk).$$

Proof. Note that any two Morse functions f, g are connected by a smooth family of smooth functions $f_t\colon M \to \mathbb{R}$, $0 \le t \le 1$ (not necessarily Morse) where the gradient-like vector fields can be extended correspondingly. Extending this family by constants, we obtain an $\mathbb{R}$-family of smooth functions $\{f_s\colon M \to \mathbb{R} | f_s = f \text{ for } s \ll 0 \text{ and } f_s = g \text{ for } s \gg 0\}$ with a family of gradient-like vector fields $\{V_s\}$. Consider the space

$$\mathcal{M}(p,q;\{f_s\}) = \left\{ x \in M \,\middle|\, \lim_{t\to+\infty} x(t) = p,\ \lim_{t\to-\infty} x(t) = q,\ \frac{dx}{dt} = V_t(x(t)) \right\},$$

of integral curves connecting critical points q of f and p of g. As in the previous section, we can regard it as a zero set of some section of a Banach bundle $W^{1,p}(\mathbb{R}, TM; p,q) \to W^{1,p}(\mathbb{R}, M; p,q)$, where the regularity question can be reduced to Fredholm property of the linearized operator

$$\frac{d}{dt} + dV_t\colon W^{1,p}(\mathbb{R}, x^*TM) \to L^p(\mathbb{R}, x^*TM),$$

and for simplicity, we choose a Riemannian metric on M so that $V_t = \nabla f_t$ and hence $dV_t = A_t$ is the Hessian metric on M (which is then symmetric). Since A_ts are symmetric, we can choose a one-parameter family of invertible matrices Φ_t so that $D_t = \Phi_t A_t \Phi_t^{-1}$ is diagonal, and we arrange the eigenvalues $\lambda_1(t), \cdots, \lambda_n(t)$ of A_t so that for $t \ll 0$ they are in increasing order. Write $\xi = (\xi_1, \cdots, \xi_n)$ and let $T_0 \in \mathbb{R}$ be a negative real number so that $f_{T_0} = f$ and $f_s \neq f$ when $s > T_0$, then we see that for a fixed $t_0 \ll T_0$, solutions to the ODE is given by

$$\xi_i(t) = e^{-t\lambda_i^- + t_0\lambda_i^-}\xi_i(t_0)$$

for $t \in \mathbb{R}$. Since $\xi_i \in W^{1,p}(\mathbb{R}, x^*TM)$, the solution ξ_i is non-zero only when $\lambda_i(t) = \lambda_i^- < 0$. Symmetrically, we see that for $t \gg 0$ such that $f_t = g$, the solution $\xi_i \in W^{1,p}(\mathbb{R}, x^*TM)$

only when $\lambda_i(t) = \lambda_i^+ > 0$. Therefore the dimension of the space of solutions is exactly the number of eigenvalues $\lambda_i(t)$ that change from negative to positive as t goes from $-\infty$ to $+\infty$, which is exactly the difference of their Morse indices, i.e. $\mathrm{Ind}(q) - \mathrm{Ind}(p)$.

Compactness for this space is the same as $\mathcal{M}(p, q)$, so we will skip the argument. Note that there're no $\mathbb{R}$-invariances anymore, so we have smooth compact 0-dimensional manifolds $\mathcal{M}(p, q; \{f_s\})$ exactly when $\mathrm{Ind}(q) = \mathrm{Ind}(p)$. Signed count of these moduli spaces provide a map

$$C_*(f; \Bbbk) \xrightarrow{\Phi_{f_s}} C_*(g; \Bbbk).$$

To show that this is a chain map, we look at the case when $\mathrm{Ind}(q) = \mathrm{Ind}(p) + 1$. In this case, the boundary in the compactification of $\mathcal{M}(p, q; \{f_s\})$ consists of broken trajectories of the form $\mathcal{M}(p, r; \{f_s\}) \times \mathcal{M}(r, q)$ and $\mathcal{M}(p, r) \times \mathcal{M}(r, q; \{f_s\})$ for some critical point r with $\mathrm{Ind}(r) = \mathrm{Ind}(p)$. This then implies that $\partial_g \circ \Phi_{f_s} = \Phi_{f_s} \circ \partial_f$ and hence Φ_{f_s} is a chain map.

Note that swapping f and g provides the other chain map $\Phi_{g_s} \colon C_*(g; \Bbbk) \to C_*(f; \Bbbk)$, and we need to verify that $\Phi_{f_s} \circ \Phi_{g_s}$ and $\Phi_{g_s} \circ \Phi_{f_s}$ are chain homotopic to the identity. To do this, consider a generic two-parameter family $F_{s,t}$ of smooth functions with $F_{s,0} = f_s \# g_s$ and $F_{s,1} \equiv f$. Then we can define a chain homotopy $K \colon C_*(f; \Bbbk) \to C_{*+1}(f; \Bbbk)$ by counting the moduli space $\mathcal{M}(p, q; \{F_{s,t}\})$ of integral curves connecting critical points q of f and p of f with respect to the family $\{F_{s,t}\}$, which is a smooth compact one-dimensional manifold when $\mathrm{Ind}(q) = \mathrm{Ind}(p) + 2$. The boundary of this one-dimensional manifold consists of broken trajectories of the form $\mathcal{M}(p, r; \{F_{s,t}\}) \times \mathcal{M}(r, q)$ and $\mathcal{M}(p, r) \times \mathcal{M}(r, q; \{F_{s,t}\})$ for some critical point r with $\mathrm{Ind}(r) = \mathrm{Ind}(p) + 1$, and hence implies that $\Phi_{f_s} \circ \Phi_{g_s} - \mathrm{Id} = \partial_f \circ K + K \circ \partial_f$. Symmetrically, we can show that $\Phi_{g_s} \circ \Phi_{f_s}$ is also chain homotopic to the identity, so that Φ_{f_s} and Φ_{g_s} are quasi-isomorphisms. $\square$

Therefore Morse homology $HM_*(M, f; \Bbbk)$ is independent of the choice of Morse-Smale pair (f, V), and hence we can simply write $HM_*(M; \Bbbk)$ for the Morse homology of M with coefficients in $\Bbbk$. Moreover, we can identify $HM_*(M; \Bbbk)$ with the singular homology $H_*(M; \Bbbk)$ of M.

4.20 Proposition. Suppose that M is a compact smooth manifold, then $HM_*(M; \Bbbk) \cong H_*(M; \Bbbk)$, where $H_*(M; \Bbbk)$ is the singular homology of M.

Proof. Pick a Morse-Smale pair (f, V) on M. Since M is compact, f must have minima $p_{n,1}, \cdots, p_{n,k_n}$ and maxima $p_{0,1}, \cdots, p_{0,k_0}$ where $n = \dim M$. The unstable manifold $W^u(p_{0,i})$ of index 0 critical points are simply points, which forms the vertices of M. For index 1 critical points $p_{1,1}, \cdots, p_{1,k_1}$, their stable manifolds $W^s(p_{1,i})$ must have index 0 critical points as their compactifications, and hence they provide edges connecting these vertices. Inductively, we see that compactifications of stable manifolds $W^s(p)$ of a critical point p of index k gives an attachment of a k-cell to a $(k-1)$-skeleton in M coming from union of stable manifolds of critical points of index $< k$. Finally, the union of stable manifolds of index n critical points form n-cells attached to some $(n-1)$-skeleton, and we see that the closure should be the whole of M, and therefore a Morse function f provides in this way a CW decomposition of M.

Now it's clear that critical points are in bijection with cells in the CW decomposition of M, and hence we get an isomorphism of graded modules $C_*(M, f, V; \Bbbk) \to C_*^{CW}(M; \Bbbk)$. The remaining task is to show that it's a chain complex, which we will omit here. See [AD14] for more details. $\square$

4.21 Corollary (Morse inequality). Let $f\colon M \to \mathbb{R}$ be a Morse function on a compact smooth manifold M, then the number of critical points of f is at least the sum of the Betti numbers of M, i.e. $\#\,\mathrm{Crit}(f) \geq \sum_k b_k(M;\mathbb{Q})$.

Proof of Theorem 4.6. Suppose $L = \Gamma(\alpha)$ is an exact Lagrangian graph in T^*Q, then $\alpha = df$ is exact for some smooth function $f\colon Q \to \mathbb{R}$, and hence the intersection points of L with the zero section Q are exactly the critical points of f. If f is Morse then Corollary 4.21 implies that $\#(L \cap Q) \geq \sum_k b_k(Q;\mathbb{Q})$. Otherwise, as all such graphs are limits of graphs of df for some Morse function f, the number of intersection points counted with multiplicities are unchanged. We therefore conclude the proof. □

26 Exercise. For $Q = \mathbb{T}^n$, compute the sum of Betti numbers of Q.

27 Exercise. For $M = \mathbb{C}P^n$, determine its singular homology $H_*(M;\mathbb{Z})$ as a $\mathbb{Z}$-module.

Hint: $\mathbb{C}P^n$ admits a Morse function with exactly $n+1$ critical points. What is that function?

9 Problem. As we have seen, Morse homology is built out from a sequence of spaces coming from trajectories between critical points of Morse functions. We should be able to get more information from the topology of these spaces than simply singular homology. In fact, there is a notion of **flow category** that can be built out from the data of Morse homology, and by taking "homology" of this flow category we can recover the Morse homology. Explain the construction of this flow category and how it recovers the Morse homology.

References: [CJS95, AB24].

10 Problem. There is an interesting A_∞-structure on the Morse complex $C_*(M,f;\Bbbk)$ for a given Morse function f, which requires a deeper look at the moduli spaces of trajectories. Explain this operadic structure and the construction of this A_∞-structure on the Morse complex.

References: [Fuk97, Hic17].

5. Arnold's Conjecture and Floer Homology

(5a) Preliminaries. Based on theorem 4.6, Arnold made the following two conjectures.

5.1 Conjecture. Let (M,ω) be a compact symplectic manifold and $L \subseteq M$ a compact Lagrangian submanifold, then for any Hamiltonian isotopy ϕ_H^t of M such that $\phi_H^1(L) \cap L$ transversely, the number of intersection points of $\phi_H^1(L)$ with L is at least the sum of Betti numbers of L, i.e. $\#(\phi_H^1(L) \cap L) \geq \sum_k b_k(L;\mathbb{Q})$.

5.2 Conjecture. Let (M,ω) be a compact symplectic manifold and $H_t\colon M \to \mathbb{R}$ an $\mathbb{S}^1$-family of time-dependent hamiltonian function on M. Assume that all fixed points of the time-one map $\phi_{H_t}^1$ are **non-degenerate**, i.e. the graph of $\phi_{H_t}^1$ intersects transversely with the diagonal in $M \times M$, then the number of fixed points of $\phi_{H_t}^1$ is at least the sum of Betti numbers of M, i.e. $\#\operatorname{Fix}(\phi_{H_t}^1) \geq \sum_k b_k(M;\mathbb{Q})$.

The conjecture of Arnold for time-dependent Hamiltonian diffeomorphisms is in fact stronger: he conjectured that the number of fixed points of $\phi_{H_t}^1$ should be at least the number of critical points of some Morse function on M. After its announcement in 1965 [Ad65], there're several works on certain special cases, and until 1988 where Floer published his seminal series of papers [Flo87, Flo88b, Flo88c, Flo88a, Flo89a, Flo89b] which resolved the conjecture for a large class of symplectic manifolds.

Floer's approach to resolve Arnold's conjecture is to construct a homology theory for Lagrangian submanifolds/Hamltonian diffeomorphisms, which mimics the Morse homology and we now call them **Floer homology**. There're technical difficulties that forces Floer to make several restrictions on the symplectic manifolds, though the difficulties were slowly overcomed by other symplectic geometers which lead to proofs of Arnold conjecture for wider and wider classes of symplectic manifolds. Some notable efforts are listed here: [Oh92, FO99, LT98, Rua99]. The Arnold conjecture for Hamiltonian diffeomorphisms was finally resolved in full generality by Abouzaid-Blumberg [AB21] for fields of positive characteristic and by Bai-Xu [BX22] for integers. The Lagrangian version is much more difficult, and it is wrong for general Lagrangian submanifolds. In this lecture, we focus mainly on Floer's original proof and don't go into the general situation, which is only about a specific problem during the construction.

5.3 Theorem (Floer). Let (M,ω) be a compact **monotone** symplectic manifold, i.e. there exists a constant $\lambda > 0$ such that $\omega(A) = \lambda c_1(A)$ for all $A \in H_2(M;\mathbb{Z})$, and $H_t\colon M \to \mathbb{R}$ an $\mathbb{S}^1$-family of time-dependent Hamiltonian function on M. Assume that all fixed points of the time-one map $\phi_{H_t}^1$ are non-degenerate, then the number of fixed points of $\phi_{H_t}^1$ is at least the sum of Betti numbers of M, i.e. $\#\operatorname{Fix}(\phi_{H_t}^1) \geq \sum_k b_k(M;\mathbb{Q})$.

5.4 Theorem (Floer). Let (M,ω) be a compact symplectic manifold and $L \subseteq M$ a compact Lagrangian submanifold such that $\pi_2(M,L) = 0$, i.e. any smooth map $u\colon (\mathbb{D}^2, \partial\mathbb{D}^2) \to (M,L)$ is null-homotopic, then for any Hamiltonian isotopy ϕ_H^t of M such that $\phi_H^1(L) \cap L$ transversely, the number of intersection points of $\phi_H^1(L)$ with L is at least the sum of Betti numbers of L, i.e. $\#(\phi_H^1(L) \cap L) \geq \sum_k b_k(L;\mathbb{Q})$.

28 Exercise. Show that $\pi_2(M,L) = 0$ implies that M is symplectically aspherical.

Let's provide a formal definition of the first Chern class.

5.5 DEFINITION. Let M be a compact smooth manifold with an almost complex structure J. The **first Chern class** of M is the Poincaré dual of the zero locus of a generic section of the complex line bundle $\det(TM, J)$, where $\det(TM, J)$ is the top wedge product of the complex vector bundle (TM, J).

Let's warm up with a proof of theorem 5.3 in the special case when H_t is independent of time.

5.6 PROPOSITION. Suppose $H\colon M \to \mathbb{R}$ is a time-independent Hamiltonian function on a compact symplectic manifold M whose time-1 periodic orbits are non-degenerate, then the number of periodic orbits of H is at least the sum of Betti numbers of M, i.e. $\#\{\text{periodic orbits of } H\} \geq \sum_k b_k(M; \mathbb{Q})$.

Proof. Note that constant periodic orbits of H are exactly the critical points of H, and they are non-degenerate iff they are non-degenerate as critical points. We then conclude the proof by applying Corollary 4.21 to H. □

29 EXERCISE. Suppose that $H_t\colon M \times \mathbb{S}^1 \to \mathbb{R}$ is a smooth time-dependent Hamiltonian function on a compact symplectic manifold M, then all its non-degenerate periodic orbits of H are isolated.

Proof. Suppose that $\{\gamma_n\colon \mathbb{S}^1 \to M\}$ is a sequence of periodic orbits of X_{H_t} that converges to a periodic orbit γ_∞, then we can pick a sequence of points $p_n \in \gamma_n$ that converges to a point $p_\infty \in \gamma_\infty$. Since γ_n is a periodic orbit, we have $\phi_{H_t}^{t_n}(p_n) = p_n$ and $\phi_{H_t}^{t_\infty}(p_\infty) = p_\infty$.

Pick a compatible almost complex structure J and consider the induced Riemannian metric g_J. We can then find unit tangent vectors $v_n \in T_{p_\infty}M$ such that the geodesic curve $\alpha_{v_n}(t)$ with initial conditions $\alpha_{v_n}(0) = p_\infty$ and $\dot\alpha_{v_n}(0) = v_n$ satisfies $\alpha_{v_n}(d(p_n, p_\infty)) = p_n$. We then get a sequence of vectors $\{v_n\} \subseteq \mathbb{S}T_pM$ which has a convergent subsequence to some unit tangent vector $v_\infty \in \mathbb{S}T_{p_\infty}M$.

Now consider the map $d\phi_{H_t}^{t_\infty}\colon T_pM \to T_pM$. By definition, we have

$$\begin{aligned} d\phi_{H_t}^{t_\infty}(v_\infty) &= \lim_{n\to\infty} d\phi_{H_t}^{t_n}(v_n) = \lim_{n\to\infty} \frac{d}{dt}\bigg|_{t=0} \phi_{H_t}^{t_n}(\alpha_{v_n}(t)) \\ &= \lim_{n\to\infty}\lim_{t\to 0} \frac{\phi_{H_t}^{t_n}(\alpha_{v_n}(t)) - p_\infty}{t} \\ &= \lim_{n\to\infty} \frac{p_n - p_\infty}{d(p_n, p_\infty)} = v_\infty, \end{aligned}$$

where starting from the second line we pick a coordinate chart near p and suppose n is sufficiently large so that p_n lies in this chart. This implies that $d\phi_{H_t}^{t_\infty}$ must have eigenvalue 1, which contradicts the non-degeneracy of γ_∞. □

30 EXERCISE. Show that 1-periodic orbits of a time-dependent Hamiltonian function $H_t\colon \mathbb{R} \times M \to \mathbb{R}$ are the same as 1-periodic orbits of another time-dependent Hamiltonian function $K_t\colon \mathbb{S}^1 \times M \to \mathbb{R}$, and hence it suffices to consider a $\mathbb{S}^1$-family of Hamiltonian functions.

Hint: Suppose $\alpha\colon [0,1] \to [0,1]$ is a monotonically increasing smooth function that is 0 of infinite order near 0 and is 1 of infinite order near 1. What is the Hamiltonian family associated to the flow $\phi_{H_t}^{\alpha(t)}$?

(5a-i) The Lagrangian Grassmannian and Maslov index. We collect here the definitions of the Lagrangian Grassmannian, the Maslov index, and the Conley-Zehnder index, which will be used throughout.

5.7 Definition. We define the **Lagrangian Grassmannian** $\mathcal{L}Gr(V)$ of a symplectic vector space (V, ω) to be the set of Lagrangian subspaces of V.

To equip $\mathcal{L}Gr(V)$ with a topology, note that it's a subspace of the Grassmannian $Gr(n, n)$ of n-dimensional subspaces of V, and hence inherits a topology from the Grassmannian. Alternatively, note that

31 Exercise. $\mathrm{Sp}(2n)$ acts transitively on $\mathcal{L}Gr(V)$, and the stabilizer of a given Lagrangian subspace $L \in \mathcal{L}Gr(V)$ can be identified with the group $\mathrm{St}(V)$ of "upper-trangular" symplectic matrices.

Proof. Given Lagrangian subspaces L_0 and L_1, we pick complementary Lagrangian subspaces K_0, K_1 so that $L_0 \oplus K_0 = L_1 \oplus K_1 = V$. Then for arbitrary choice of basis on L_i and K_i, the linear map $S\colon V \to V$ sending corresponding basis of L_0 to L_1 and K_0 to K_1 is a symplectomorphism.

Identifying V with $\mathbb{R}^{2n}$, we can consider the Lagrangian subspace $L_0 = \mathbb{R}^n \times \{0\} \subseteq V$. Then any symplectic matrix can be written as the block form

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and we have shown in Lemma 3.2 what these matrices should satisfy. Sending $\mathbb{R}^n \times \{0\}$ to itself implies that $Cv = 0$ for any vector $v \in \mathbb{R}^n$, and hence $C = 0$, so the stabilizer of $\mathbb{R}^n \times \{0\}$ is exactly the group of upper-triangular symplectic matrices. $\square$

The exercise shows that $\mathcal{L}Gr(V, \omega)$ is a smooth manifold, and let's investigate its tangent bundle. Fix a Lagrangian subspace $L_0 \in \mathcal{L}Gr(V, \omega)$ and a Lagrangian subspace K_0 complement to L_0, we can identify an open neighbourhood of L_0 in $\mathcal{L}Gr(V, \omega)$ with a subspace of the space of linear maps $\hom(L_0, K_0)$ from L_0 to K_0: for each $A \in \hom(L_0, K_0)$, the corresponding Lagrangian subspace in $\mathcal{L}Gr(V)$ is the graph

$$L_A = \{(v, A(v))\colon v \in L_0\}.$$

32 Exercise. Prove that L_A is a Lagrangian subspace of V iff A is symmetric.

In particular, we can identify locally any curve $L(t)$ of Lagrangian subspaces with a curve $A(t)$ of symmetric matrices, and hence the tangent space $v \in T_{L_0}\mathcal{L}Gr(V)$ of $\mathcal{L}Gr(V)$ at a Lagrangian subspace L_0 can be identified with the space of symmetric bilinear forms on L_0.

Fix $L_0 \in \mathcal{L}Gr(V)$, then the Lagrangian Grassmannian $\mathcal{L}Gr(V)$ also admits a stratification

$$\mathcal{L}Gr(V) = \mathcal{L}Gr(V, L_0)_0 \supseteq \mathcal{L}Gr(V, L_0)_1 \supseteq \cdots \supseteq \mathcal{L}Gr(V, L_0)_n = \{L_0\},$$

relative to L_0, by the dimension of the intersection $L \cap L_0$. A tangent vector $v \in T_{L_0}\mathcal{L}Gr(V)$ belongs to $T_{L_0}\mathcal{L}Gr(V, L_0)_k$ if the symmetric bilinear form $a\colon L_0 \to K_0$ corresponding to v has kernel of dimension at least k.

5.8 Definition. Fix a curve $L\colon [a, b] \to \mathcal{L}Gr(V)$ of Lagrangian subspaces. A **crossing** of L relative to L_0 is a point $t \in [a, b]$ such that $L(t_0) \in \overline{\mathcal{L}Gr(V, L_0)_1}$. A crossing t is **regular** if the corresponding tangent vector $v \in T_{L(t)}\mathcal{L}Gr(V)$ is non-singular when restricted to $L(t_0) \cap L_0$, and is **simple** if furthermore $L(t_0) \in \mathcal{L}Gr(V, L_0)_1$.

Note that simplicity is the same as saying L is transversal to $\overline{\mathcal{L}Gr(V, \omega)_1}$. Given bilinear form $A\colon W \otimes W \to \mathbb{R}$ on a real vector space W, we define its **signature** $\operatorname{sgn}(A)$ to be the number of positive eigenvalues of A minus the number of negative eigenvalues of A.

5.9 Definition. The **Maslov index** of a path $L(t)$ with respect to a given Lagrangian subspace L_0 is defined to be the sum of signatures

$$\mu(L, L_0) = \frac{1}{2}\left(\operatorname{sgn} A(b)|_{L_0 \cap L(a)} + \operatorname{sgn} A(a)|_{L_0 \cap L(b)}\right) + \sum_{a<t<b} \operatorname{sgn} A(t)|_{L_0 \cap L(t)}.$$

For convenience, we write $A(t) =: \Gamma(L'(t), L_0; t)$ to be the **crossing form** of $L(t)$ at t with respect to L_0.

Note that if $L(t)$ does not pass through $\overline{\mathcal{L}Gr(V, L_0)_1}$, then $L_0 \cap L(t) = \{0\}$ and hence $\operatorname{sgn} A(t)|_{L_0 \cap L(t)} = 0$. Since regular crossings are clearly isolated, this sum is well-defined and finite.

(5a-ii) The Conley-Zehnder index. The index $\mu_{CZ}(\gamma, [\Gamma])$ associated to a contractible Hamiltonian loop is called the **Conley-Zehnder index** of $(\gamma, [\Gamma])$, which is defined as follows: a map $\Gamma\colon \overline{\mathbb{D}}^2 \to M$ induces a symplectic vector bundle $\Gamma^*TM \to \overline{\mathbb{D}}^2$ over the closed disk $\overline{\mathbb{D}}^2$, where the linearized Hamiltonian flow $d\phi^t_{H_t}$ on $\partial\overline{\mathbb{D}}$ induces a one-parameter family of linear symplectomorphisms $d\phi^t_{H_t}|_{\gamma(t)}\colon T_{\gamma(0)}M \to T_{\gamma(t)}M$, and under a chosen trivialization $\Gamma^*TM \cong \overline{\mathbb{D}} \times \mathbb{R}^{2n}$, we can identify $d\phi^t_{H_t}$ as a smooth family of symplectic matrices $S\colon [0, 1] \to \operatorname{Sp}(2n)$ with $S(0) = \operatorname{Id}$.

Consider the following stratification

$$\operatorname{Sp}(2n) \supseteq \operatorname{Sp}_1(2n) \supseteq \cdots \supseteq \operatorname{Sp}_n(2n) = \{\operatorname{Id}\}$$

of $\operatorname{Sp}(2n)$ by the number of eigenvalues equal to 1, then it's clear that $\operatorname{Sp}_k(2n)$ is a submanifold of $\operatorname{Sp}(2n)$ of codimension $k(k+1)/2$. Roughly speaking, the Conley-Zehnder index $\mu_{CZ}(\gamma, [\Gamma])$ should be the intersection number of the path $S\colon [0, 1] \to \operatorname{Sp}(2n)$ with the compactification $\overline{\operatorname{Sp}_1(2n)}$, which is called the **Maslov cycle**.

This intersection is, however, non-generic, and hence we have to count intersection numbers with multiplicities.

5.10 Definition. Let $\Psi\colon [0, 1] \to \operatorname{Sp}(2n)$ be a path from Id to a non-degenerate symplectic matrix $\Psi(1)$, then the **Conley-Zehnder index** is the Maslov index

$$\mu_{CZ}(\Psi) = \mu(\operatorname{Gr}(\Psi), \Delta)$$

in the product symplectic vector space $(V \oplus V, \omega \oplus -\omega)$.

It's difficult in general to compute the index, but we can do this in special cases, for example for non-degenerate constant orbits of a time-independent Hamiltonian function H. Pick a Darboux neighbourhood $U(p)$ of this non-degenerate constant orbit $p \in M$, so that we can write

$$X_H = \sum_{i=1}^{n} -\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} + \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i},$$

which is 0 at 0. Suppose that $\{x_s\}$ is a curve in $U(p)$ with $x_0 = 0$ and $\dot{x}_0 = v$, then we can compute that

$$\begin{aligned}
(d\phi_H^t)_0(v) &= \lim_{s\to 0} \frac{\phi_H^t(x_s) - \phi_H^t(0)}{s} \\
&= \lim_{s\to 0} \left(\frac{1}{s}\int_0^t (X_H(x_s) - X_H(x_0))d\tau + \frac{1}{s}\int_0^t x_s' d\tau \right) \\
&= v + t(dX_H)_0(v),
\end{aligned}$$

where dX_H is exactly the Hessian matrix of H, and is hence non-degenerate by assumption. Therefore the curve $(d\phi_H^t)_0$ passes through $\overline{\mathrm{Sp}_1(2n)}$ only when $t = 0$, whose derivative with respect to t is the symmetric matrix $dX_H \colon T_0\mathbb{R}^{2n} \to T_0\mathbb{R}^{2n}$, which is exactly the symmetric bilinear form on $\Delta \subseteq V \oplus V$ that contributes to the computation of the Conley-Zehnder index. We therefore conclude that

33 Exercise. $\mu_{CZ}(p) = \frac{1}{2}\operatorname{sgn} d^2H_p = n - \mathrm{Ind}_p(p)$ for a non-degenerate constant orbit p of a time-independent Hamiltonian function H.

(5a-iii) Dependence on cappings and the minimal Chern number. The Conley-Zehnder index determines the dimension of the moduli space, but it depends on the choice of cappings Γ of a given Hamiltonian loop γ. Suppose Γ_1, Γ_2 are two cappings of γ, then we obtain two different trivializations of the symplectic vector bundle $\gamma^*TM \to \mathbb{S}^1$, and the difference between them provides a loop of symplectic matirces $\mathbb{S}^1 \to \mathrm{Sp}(2n)$, if we fix a base point $1 \in \mathbb{S}^1$. Recall the polar decomposition formula for special linear matrices:

5.11 Proposition ([Hal03, Proposition 2.19]). Any $S \in GL(n, \mathbb{R})$ admits a unique decomposition

$$S = Ue^X$$

where $U \in O(n)$ is an orthogonal matrix and X is real and symmetric.

Proof. Note that the matrix S^TS is always positive definite and self-adjoint, so we can take the square root matrix $\sqrt{S^TS}$ which is still positive definite, self-adjoint, and has determinant 1. We can therefore find a symmetric matrix X such that $\sqrt{S^TS} = e^X$. Now let $U = Se^{-X}$, then we can verify that

$$UU^T = Se^{-X}e^{-X}S^T = S(S^TS)^{-1}S^T = \mathrm{Id} = U^TU,$$

which implies that U is orthogonal. $\square$

34 Exercise. Show that $\mathrm{Sp}(2n) \cap O(2n) = U(n)$, where $U(n)$ is the unitary group of $n \times n$ complex matrices by identifying $\mathbb{R}^{2n} \cong \mathbb{C}^n$.

35 Exercise. Show that for a symplectic matrix $S \in Sp(2n) \subseteq GL(2n, \mathbb{R})$, under the polar decomposition

$$S = Ue^X$$

where U is unitary and X should furthermore satisfy $X\omega + \omega X = 0$.

In particular, $\mathrm{Sp}(2n)$ is homotopic equivalent to $U(n)$, and hence

$$\pi_1(\mathrm{Sp}(2n), 1) \cong \pi_1(U(n), 1) \cong \mathbb{Z}$$

is determined by (an extension of) the determinant function $\det\colon U(n) \to \mathbb{S}^1$.

On the other hand, we can glue the cappings Γ_1 and Γ_2 to obtain a symplectic vector bundle over the sphere $\Gamma_1 \# \Gamma_2 \colon \mathbb{S}^2 \to M$, which is trivial iff there exists a global non-vanishing section over $\mathbb{S}^2$. Consider a triangulation of $\mathbb{S}^2$ where the equator is the 1-skeleton with two 2-cells attached to it. Since $\mathrm{Sp}(2n)$ is connected it's easy to construct a section on the 1-skeleton, and the obstruction to extend it to the 2-cells is exactly given by the representative of the 1-skeleton $\mathbb{S}^1 \hookrightarrow \mathbb{S}^2$ in $\pi_1(\mathrm{Sp}(2n)) \cong \mathbb{Z}$, and hence this representative in $\mathbb{Z} \cong \pi_1(\mathrm{Sp}(2n), 1)$ is the same as $c_1([\Gamma_1 \# \Gamma_2^{-1}]) =: c_1(A)$.

We want to compare the Conley-Zehnder index of this loop $\mathbb{S}^1 \xrightarrow{\Psi} \mathrm{Sp}(2n)$ through Id with the Chern number $c_1(A)$. We will not prove this fact, but it will turn out that $\mu_{CZ}(\Psi) = 2c_1(A)$, and therefore the Conley-Zehnder index of a Hamiltonian loop γ is well-defined up to twisting by $2c_1(A)$ for $A \in H_2(M; \mathbb{Z})$.

5.12 Definition. Given a symplectic manifold (M, ω) with $c_1 \colon H_2(M; \mathbb{Z}) \to \mathbb{Z}$, we define the **minimal Chern number** $mC(M, \omega)$ of M to be the minimum of $|c_1(A)|$ over all elements of $A \in H_2(M; \mathbb{Z})$ such that $c_1(A) \neq 0$.

(5b) Hamiltonian Floer homology. The main idea of the proof of Theorem 5.3 is to find a proper space so that 1-periodic orbits of H_t can be regarded as critical points of some smooth function, and then to mimic the construction of Morse homology. We now carry this out in detail.

(5b-i) The action functional and Floer's equation. Recall from physics that for a loop $\gamma \colon \mathbb{S}^1 \to T^*Q$, the **action** of γ is defined by

$$\mathcal{A}(\gamma) = \int_{\mathbb{S}^1} \lambda_{\mathrm{tau}} - \int_0^1 H_t(\gamma(t))\, dt.$$

For a general compact symplectic manifold (M, ω), the action functional is not well-defined on an arbitrary orbit γ, due to the problem that a symplectic structure is not exact. To fix this, we consider an abelian cover of the space of contractible loops

$$\widetilde{\mathcal{L}}_c(M) = \left\{ (\gamma, [u]) \Big|_{u_1 \simeq u_2 \Leftrightarrow u_1 \text{ is homotopic to } u_2 \text{ rel } \partial\mathbb{D}^2}^{u \in C^\infty(\mathbb{D}^2, M), \gamma = u(\partial\mathbb{D}^2)} \right\},$$

so for each $(\gamma, [u]) \in \widetilde{\mathcal{L}}_c(M)$, the **action functional** of $(\gamma, [u])$ is given by the integral

$$\mathcal{A}(\gamma, [u]) = \int_{\mathbb{D}^2} u^*\omega - \int_0^1 H_t(\gamma(t))\, dt.$$

This action functional defines a differentiable function on the infinite-dimensional Fréchet manifold $\widetilde{\mathcal{L}}_c(M)$. To compute its differential, note that a curve in $\widetilde{\mathcal{L}}_c(M)$ is a map $\Gamma\colon (-\varepsilon, \varepsilon) \times \mathbb{S}^1 \to M$, so that for a given vector field $\xi \in T_\gamma(\widetilde{\mathcal{L}}_c(M)) = C^\infty(\mathbb{S}^1, \gamma^* TM)$, we can compute the differential as

$$d\mathcal{A}(\xi) = \frac{\partial}{\partial s}\bigg|_{s=0} \mathcal{A}(\Gamma(s,-)) = \int_{\mathbb{D}^2} \omega(\xi, \dot{\gamma}) - \int_0^1 dH_t(\xi)\, dt = \int_{\mathbb{D}^2} \omega(\xi, \dot{\gamma} - X_H).$$

For a chosen $J \in \mathcal{J}_\omega(M)$, we have a natural L^2-metric on $\widetilde{\mathcal{L}}_c(M)$ given by

$$\langle \xi, \eta \rangle_{L^2} = \int_0^1 g_J(\xi(t), \eta(t))\, dt = \int_0^1 \omega(\xi(t), J\eta(t))\, dt,$$

so we see that the **gradient vector field** of $\mathcal{A}$ under g_J is

$$\langle \nabla \mathcal{A}, \xi \rangle_{L^2} = \int_{\mathbb{D}^2} \langle \xi, J(X_H - \dot{\gamma}) \rangle,$$

i.e. $\nabla \mathcal{A} = J(X_H - \dot{\gamma})$. Therefore critical loops of $\mathcal{A}$ are exactly the 1-periodic orbits of H, and the **negative gradient flow** of $\mathcal{A}$ from γ_+ to γ_- is given by a map $u\colon \mathbb{R} \times \mathbb{S}^1 \to M$ satisfying the equation

$$\begin{cases} \dfrac{\partial u}{\partial s} + J\left(\dfrac{\partial u}{\partial t} - X_H\right) = 0 \\ \lim\limits_{s \to +\infty} u(s,t) = \gamma_+(t) \\ \lim\limits_{s \to -\infty} u(s,t) = \gamma_-(t) \end{cases} \tag{5}$$

This is called the **Floer's equation** for a given time-dependent family of Hamiltonian functions. Note that this equation is a perturbation of the non-linear Cauchy-Riemann equation $\bar{\partial}_J u = 0$ by the Hamiltonian vector field X_H. As the perturbation term has order 0, it is a **compact operator** and hence does not affect the major properties of the differential operator, meaning that regularity and Fredholm property of the linearized operator still holds.

As in the construction of Morse homology, we want to understand the space of trajectories

$$\widehat{\mathcal{M}}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; J, A) = \{u\colon \mathbb{R} \times \mathbb{S}^1 \to M | (5),\ [\Gamma_- \# u \# \Gamma_+] = A\}$$

for given pairs $(\gamma_-, [\Gamma_-])$ and $(\gamma_+, [\Gamma_+])$ of Hamiltonian periodic orbits of H_t, where $A \in H_2(M; \mathbb{Z})$ is a spherical class and $\Gamma_- \# u \# \Gamma_+$ is the sphere given by gluing to u the two cappings Γ_- and Γ_+.

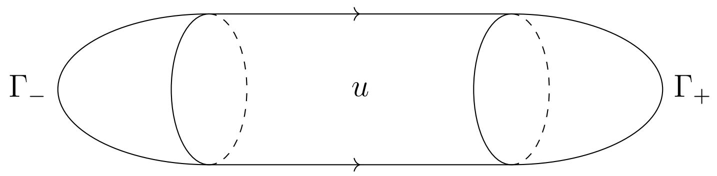

Figure 5: Trajectory u and cappings.

Recall the definition of energy of a curve $u\colon \mathbb{R} \times \mathbb{S}^1 \to M$ is given by

$$E(u) = \int_{\mathbb{R} \times \mathbb{S}^1} |du|^2 ds\, dt = 2 \int_{\mathbb{R} \times \mathbb{S}^1} u^* \omega + \int_{\mathbb{R} \times \mathbb{S}^1} |X_H|^2 ds\, dt.$$

On the other hand, we have

$$\mathcal{A}(\gamma_+, [\Gamma_+]) - \mathcal{A}(\gamma_-, [\Gamma_-]) = \omega(A) + \int_0^1 |X_H|^2 ds\, dt,$$

where it's clear that $\omega(A) \geq \int_{\mathbb{R}\times\mathbb{S}^1} u^*\omega$. Therefore $E(u) < \infty$ for all such u.

(5b-ii) Regularity. The transversality framework for the Floer equation is the same as the non-linear Cauchy-Riemann equation, where we consider the completion of $C^\infty(\mathbb{S}^1 \times \mathbb{R}, M; \gamma_-, \gamma_+; A)$ by the $W^{1,p}$-norm, and regard the space of trajectories as solutions to a differential operator on the tangent bundle $T^{0,1}W^{1,p}(\mathbb{S}^1 \times \mathbb{R}, M; \gamma_-, \gamma_+; A) \to W^{1,p}(\mathbb{S}^1 \times \mathbb{R}, M; \gamma_-, \gamma_+; A)$. The regularity question is about surjectivity of the linearized Floer operator

$$D_{u,J,H_t}(\xi) := \bar{\partial}_J \xi + \nabla_\xi J \left(\frac{\partial u}{\partial t} - X_{H_t} \right) - J \nabla_\xi X_{H_t},$$

defined on the space of $W^{1,p}$-sections $W^{1,p}(\mathbb{R}\times\mathbb{S}^1, u^*TM; \gamma_-, \gamma_+; A)$ with given asymptotic conditions. We hope to achieve surjectivity of D_{u,J,H_t} for generic choice of the data $(J, H_t) \in \mathcal{J}_\omega \times \mathcal{H}$, where $\mathcal{H}$ is the space of time-dependent Hamiltonian functions on M. In this case, a stronger result can be achieved: for fixed $J \in \mathcal{J}_\omega(M)$, we can find a residual subset $\mathcal{H}_J^{reg} \subseteq \mathcal{H}$ to achieve regularity. This is encoded in the following theorem:

5.13 Theorem. Let (M, ω) be a compact symplectic manifold and $J \in \mathcal{J}_\omega(M)$ a compatible almost complex structure, then the following holds.

(i) the linearized operator

$$\begin{aligned} Ds_{(u,J,H_t)} \colon T_u W^{1,p}(\mathbb{R} \times \mathbb{S}^1, M; \gamma_-, \gamma_+; A) \times T_{H_t}\mathcal{H} &\to L^p(\mathbb{R} \times \mathbb{S}^1, u^*TM) \\ (Y, h) &\mapsto D_{u,J,H_t}(Y) + \nabla h(u) \end{aligned}$$

is split surjective.

(ii) For every non-degenerate Hamiltonian $\{H_t\}$, every $J \in \mathcal{J}_\omega(M)$ ω-compatible and every $u \in \mathcal{M}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; J, A)$, the linearized operator D_{u,J,H_t} is Fredholm of index $\mu_{CZ}(\gamma_+, [\Gamma_+]) - \mu_{CZ}(\gamma_-, [\Gamma_-])$.

(iii) The projection $d\pi \colon T_u W^{1,p}(\mathbb{R} \times \mathbb{S}^1, M; \gamma_-, \gamma_+; A) \times T_{H_t}\mathcal{H} \to T_{H_t}\mathcal{H}$ is Fredholm.

(5b-iii) Compactness. As in the discussion of Morse homology 4, we want to quotient the moduli space $\widehat{\mathcal{M}}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; J, A)$ by the $\mathbb{R}$-action given by translation in the s-direction, so that we get the unparametrized moduli space of trajectories

$$\mathcal{M}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; J, A) = \widehat{\mathcal{M}}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; J, A)/\mathbb{R}.$$

In particular, when $\mu_{CZ}(\gamma_+, [\Gamma_+]) - \mu_{CZ}(\gamma_-, [\Gamma_-]) = 1$, the moduli space $\mathcal{M}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; J, A)$ is a 0-dimensional manifold, and we want to count the number of trajectories appearing in this moduli space. This requires again compactness of this moduli space, which will now be closer to the Uhlenbeck-Gromov compactness theorem.

5.14 Theorem. Let $\{u_n\}$ be a sequence of trajectories in $\widehat{\mathcal{M}}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; J, A)$, then in a possible limit the following situation will arise:

- Bubbles or trees of bubbles appear at some of the points in $\mathbb{R} \times \mathbb{S}^1$, just as what happens in the Uhlenbeck-Gromov compactness theorem.
- The trajectory $\mathbb{R} \times \mathbb{S}^1$ breaks into several pieces, each of which connects a pair of Hamiltonian loops with decreasing Conley-Zehnder indices in order.

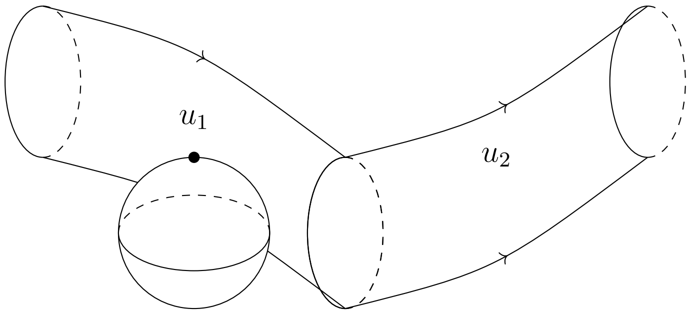

Figure 6: Possible breakings of a trajectory.

Note that when a bubble sphere arises, the dimension of the moduli space of the main component has to drop dimension by $2c_1(A)$. Recall from Morse theory that we only cares about moduli spaces of dimension 0 and 1, so if $2c_1(A) > 1$, i.e. $c_1(A) > 0$, then the bubble will not contribute to the count of trajectories. This is guaranteed by monotonicity, i.e. $c_1(A) = \lambda\omega(A) > 0$ for all non-constant holomorphic spheres representing class A.

(5b-iv) Gluing. In addition to compactness, as in the discussion of Morse complex 4, we need to understand how the compactified strata gets attached to the main stratum of the moduli space. This is referred to as **gluing** of moduli spaces (of pseudo-holomorphic curves).

We will not go into the details as it involves hardcore functional analysis, but let's just briefly explain the ideas: given a broken trajectory $u_1 \# u_2$, we can construct a smooth strip $u_s \colon S \times \mathbb{R} \to M$ which coincides with u_1 and u_2 near the two ends of the strip, and is a smoothing of $u_1(S \times \mathbb{R}) \cup u_2(S \times \mathbb{R})$ in a neighbourhood of the connecting critical chord. This construction clearly provides us a smooth family approaching $u_1 \# u_2$, and now we want to perturb it to achieve J-holomorphicity. This is accomplished by an infinite-dimensional version of implicit function theorem. As a consequence, we conclude that

5.15 Theorem. Suppose (M, ω) is monotone, then for generic choice of H_t, the moduli space of trajectories

$$\mathcal{M}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; J, A)$$

of dimension ≤ 1 can be compactified into a smooth manifold with boundary consisting of broken trajectories

$$\mathcal{M}(\gamma_-, [\Gamma_-], \gamma_0, [\Gamma_0]; J, A) \times \mathcal{M}(\gamma_0, [\Gamma_0], \gamma_+, [\Gamma_+]; J, A),$$

where $\mu_{CZ}(\gamma_+, [\Gamma_+]) - \mu_{CZ}(\gamma_0, [\Gamma_0]) = 1$ and $\mu_{CZ}(\gamma_0, [\Gamma_0]) - \mu_{CZ}(\gamma_-, [\Gamma_-]) = 1$.

(5b-v) Orientation. In order to achieve arbitrary ring coefficients, as in section 4, we need to equip the moduli space of trajectories and broken trajectories with coherent orientations, such that the orientation assigned on the moduli space of broken trajectories coincides with the boundary orientation.

5.16 Definition. Determinant line of an infinite-dimensional vector space V do not in general make sense, but for a Fredholm operator $D\colon V \to W$, we can define its **determinant line** $\det D$ to be the tensor product

$$\det D = \bigwedge^{\text{top}} \ker D \otimes \bigwedge^{\text{top}} (\operatorname{coker} D)^{\vee}.$$

For each moduli space $\mathcal{M}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; J, A)$, recall that they are zero locus of a section $s\colon C^{\infty}(\mathbb{R}\times\mathbb{S}^1, M; \gamma_-, \gamma_+; A) \to T^{0,1}C^{\infty}(\mathbb{R}\times\mathbb{S}^1, M; \gamma_-, \gamma_+; A)$, where the linearization $D_{u,H_t,J}$ is Fredholm and surjective, and hence we have a natural determinant line bundle $\det D = \det\ker D_{u,H_t,J}$ over the moduli space $\mathcal{M}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; J, A)$. To get a coherent orientation, we also need to assign natural orientations on the pairs $(\gamma, [\Gamma])$ of Hamiltonian loops and their cappings.

The capping $\Gamma\colon \overline{\mathbb{D}}^2 \to M$ provides a trivialization of the symplectic vector bundle $\Gamma^*TM \cong \overline{\mathbb{D}}^2 \times \mathbb{R}^{2n}$, and we can extend it to a symplectic vector bundle $\mathbb{C}\times\mathbb{R}^{2n}$ by attaching a trivial copy of the product $\mathbb{S}^1 \times [1, +\infty) \times \mathbb{R}^{2n}$ to it. Let $B_z\colon \mathbb{C} \to \operatorname{End}(\mathbb{R}^{2n})$ be a 2-parameter family of matrices such that $B_z|_{\mathbb{S}^1\times[1,+\infty)} = X_{H_t}$ is the vector field X_{H_t}, and let $I\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be an ω-compatible complex structure on the bundle $\mathbb{C}\times\mathbb{R}^{2n}$ which is an extension of the pull-back complex structure from $\Gamma\colon \overline{\mathbb{D}}^2 \to M$.

We can then define a partial differential operator

$$\begin{aligned} D_{B,I}\colon W^{1,p}(\mathbb{C}, \mathbb{R}^{2n}) &\to L^p(\mathbb{C}, \mathbb{R}^{2n}) \\ \xi &\mapsto \partial_s\xi + I(\partial_t\xi - B_z\xi) \end{aligned}$$

whose solutions are sections of the symplectic vector bundle $\mathbb{C}\times\mathbb{R}^{2n}$ which are asymptotic to the Hamiltonian loop γ at infinity. It's standard that $D_{B,I}$ is Fredholm, so we have a well-defined determinant line bundle $\det D_{B,I}$ over the whole of $\mathbb{C}$. We regard this as the determinant line bundle associated to the pair $(\gamma, [\Gamma])$.

With a similar argument as in Morse homology but with stable manifolds replaced by kernels and cokernels of the Floer operator, we can show that this construction provides for each pair $((\gamma_-, [\Gamma_-]), (\gamma_+, [\Gamma_+]))$ a canonical isomorphism

$$\det D_{u,H_t,J} \cong \det D_{B_-,I_-} \otimes \det D^{\vee}_{B_+,I_+},$$

so that for an arbitrary choice of orientations on each pair $(\gamma, [\Gamma])$, we get corresponding signs on each of the trajectories in $\mathcal{M}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; J, A)$. See [Abo15] for more details.

(5b-vi) The Floer complex. With all these preparations we are now ready to construct the Floer complex. For a generic non-degenerate $\mathbb{S}^1$-family of hamiltonian functions $\{H_t\}$ on a monotone symplectic manifold (M, ω), we define the **Floer complex**

$$CF_*(M; H_t; \Bbbk) = \bigoplus_{\gamma\in\mathcal{P}_c(H)} \Lambda_{\Bbbk}\langle\gamma\rangle[-\mu_{CZ}(\gamma, [\Gamma])]$$

to be the $\mathbb{Z}/2mC(M,\omega)\mathbb{Z}$-graded free $\Lambda_{\Bbbk}$-module generated by contractible Hamiltonian orbits γ, where differentials are given by

$$\partial(\gamma) = \sum_{\substack{\beta\in\mathcal{P}_c(H)\\ \mu_{CZ}(\beta,[\Gamma'])=\mu_{CZ}(\gamma,[\Gamma])-1}} \sum_{A\in H_2(M;\mathbb{Z})} \#\overline{\mathcal{M}}(\gamma, [\Gamma], \beta, [\Gamma']; J, A) T^{\omega(A)}\cdot\beta.$$

From the gluing and orientation theorems it follows that $\partial^2 = 0$, so the **Hamiltonian Floer homology**

$$HF_*(M; H_t; \Bbbk) = H_*(CF_*(M; H_t; \Bbbk), \partial)$$

is well-defined.

(5b-vii) Hamiltonian invariance. Note that the Hamiltonian Floer homology, in its definition, relies on a choice of the one-parameter family of hamiltonian functions H_t and an almost complex structure $J \in \mathcal{J}_\omega^{reg}$. However, we will show in this subsection that the Floer homology is independent of the choices, and hence provides an invariant of the symplectic manifold (M, ω).

For this reason, we call a pair $\mathcal{F} = (H_t, J_t)$ a **Floer datum** for the symplectic manifold (M, ω). The space of Floer data is always connected, so any two generic Floer data $\mathcal{F}_0 = (H_t^0, J_t^0)$ and $\mathcal{F}_1 = (H_t^1, J_t^1)$ can be connected by a smooth path $\{\mathcal{F}_s\}_{0 \leq s \leq 1}$ of Floer data.

5.17 Proposition. For given two generic Floer data $\mathcal{F}_0$, $\mathcal{F}_1$, there exists a canonical chain map

$$CF_*(M; H_t^0, J_t^0; \Lambda_\Bbbk) \to CF_*(M; H_t^1, J_t^1; \Lambda_\Bbbk)$$

that induces an isomorphism on homology.

For this reason, we can simply write $HF_*(M; \mathcal{F}; \Lambda_\Bbbk) = HF_*(M; \Lambda_\Bbbk)$.

Proof. The procedure is the same as in section 4, when we show that Morse homology does not depend on the choice of Morse functions. We consider the moduli space of trajectories

$$\mathcal{M}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; \{\mathcal{F}_s\}, A) = \left\{ u\colon \mathbb{R} \times \mathbb{S}^1 \to M \,\middle|\, \begin{array}{c} \partial_s u + J_{s,t}(\partial_t u - X_{H_{s,t}}) = 0 \\ (5),\ [\Gamma_- \# u \# \Gamma_+] = A \end{array} \right\},$$

which breaks the $\mathbb{R}$-symmetry and similar analytic properties, so that for generic choice of the family $\{\mathcal{F}_s\}$ the moduli space $\mathcal{M}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; \{\mathcal{F}_s\}, A)$ is a smooth manifold of dimension $\mu_{CZ}(\gamma_+, [\Gamma_+]) - \mu_{CZ}(\gamma_-, [\Gamma_-])$. Uhlenbeck-Gromov compactness still works in this case, where we can see the possible degenerations are of the form

$$\mathcal{M}(\gamma_-, [\Gamma_-], \gamma_0, [\Gamma_0]; \mathcal{F}_0, A_0) \times \mathcal{M}(\gamma_0, [\Gamma_0], \gamma_+, [\Gamma_+]; \{\mathcal{F}_s\}, A - A_0)$$

or

$$\mathcal{M}(\gamma_-, [\Gamma_-], \gamma_0, [\Gamma_0]; \{\mathcal{F}_s\}, A - A_1) \times \mathcal{M}(\gamma_0, [\Gamma_0], \gamma_+, [\Gamma_+]; \mathcal{F}_1, A_1)$$

when the moduli space has dimension ≤ 1.

In particular, when $\mu_{CZ}(\gamma_+, [\Gamma_+]) - \mu_{CZ}(\gamma_-, [\Gamma_-]) = 0$, the moduli space is compact and we can count the number of trajectories in $\mathcal{M}(\gamma_-, [\Gamma_-], \gamma_+, [\Gamma_+]; \{\mathcal{F}_s\}, A)$ to define a chain map

$$CF_*(M; H_t^0, J_t^0; \Lambda_\Bbbk) \to CF_*(M; H_t^1, J_t^1; \Lambda_\Bbbk).$$

When $\mu_{CZ}(\gamma_+, [\Gamma_+]) - \mu_{CZ}(\gamma_-, [\Gamma_-]) = 1$, the moduli space is a compact 1-dimensional manifold with boundary consisting of broken trajectories, which gives the chain map relation, and hence this map descends to a map of homology. By reversing the family we get an opposite chain map

$$CF_*(M; H_t^1, J_t^1; \Lambda_\Bbbk) \to CF_*(M; H_t^0, J_t^0; \Lambda_\Bbbk),$$

and by looking at a two-parameter family of Floer data connecting the identity family to the composition of the two families, we can show that the composition of these two chain maps is homotopic to the identity, and hence induces an isomorphism on homology. □

(5b-viii) Proof of Arnold's conjecture for Hamiltonian diffeomorphisms. Since Hamiltonian invariance we can simply pick H to be time-independent, though there're still questions concerning whether all 1-periodic orbits of H are constant. This can be achieved once H is sufficiently small.

5.18 PROPOSITION. Let $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function and we identify X_H with a vector field on $\mathbb{R}^{2n}$. If $\|dX_H\| < 2\pi$ where the norm is taken with respect to the standard symplectic and complex structure on $\mathbb{R}^{2n}$, then the only solutions of period 1 of the Hamiltonian system associated with H are the constant solutions, i.e. critical points.

Proof. For each curve $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^{2n}$, consider its Fourier expansion

$$\gamma(t) = \sum_{k\in\mathbb{Z}} e^{2\pi i k t}\gamma_k, \quad \dot\gamma(t) = \sum_{k\in\mathbb{Z}} 2\pi i k e^{2\pi i k t}\gamma_k.$$

Parseval's identity implies that

$$\|\ddot\gamma\|_{L^2}^2 = \sum 4\pi^2 k^2 |\dot\gamma_k|^2 \geq 4\pi^2 \sum_{k\neq 0} |\dot\gamma_k|^2 = 4\pi^2 \|\dot\gamma\|_{L^2}^2.$$

We therefore have

$$\|\dot\gamma\|_{L^2}^2 \leq \frac{1}{2\pi}\|\ddot\gamma\|_{L^2}.$$

On the other hand, as $\ddot\gamma = (dX_H)_\gamma \cdot \dot\gamma$, the condition implies that

$$\|\ddot\gamma\|_{L^2} < 2\pi\|\dot\gamma\|_{L^2} \text{ if } \dot\gamma \neq 0,$$

and hence we must have $\dot\gamma = 0$ so that γ is constant. □

5.19 PROPOSITION. Suppose that (M,ω) is a compact symplectic manifold and $H\colon M \to \mathbb{R}$ is a smooth function that is C^2-sufficiently small, then the only solutions of period 1 of the Hamiltonian system associated with H are the constant solutions.

Proof. First note that if $\gamma\colon \mathbb{S}^1 \to M$ is a 1-periodic solution of the Hamiltonian system associated with H, then we have the inequality

$$d(\gamma(t),\gamma(0)) \leq \int_0^t \|X_H(\gamma(s))\| ds \leq t \cdot \sup_{x\in M} \|X_H(x)\|,$$

so after picking a Darboux cover of M, if H is C^1-sufficiently small then all 1-periodic orbits of H are contained in a single Darboux chart, and since H is furthermore C^2-sufficiently small, we can apply the previous proposition to conclude that all 1-periodic orbits of H are constant. □

The two propositions allow us to choose a smooth Morse function H which is C^2-sufficiently small (by simply rescaling any given Morse functions) so that we get a bijective correspondence between generators of $CF_*(M;H;\Lambda_\Bbbk)$ and $CM_*(M;H;\Lambda_\Bbbk)$. This is not enough: we still need to understand the relation between their differentials.

Proof of Theorem 5.3. Suppose that $(H,\nabla H)$ is a Morse-Smale pair, then we see that the Floer equation for trajectories is given by

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = \nabla H(u),$$

which is furthermore independent of the t-variable. Therefore we must have an $\mathbb{S}^1$-action on the moduli space of trajectories, which however should not provide an extra dimension. This implies that the $\mathbb{S}^1$-action should be trivial and the trajectories should have image that is the same as the gradient trajectory of H. This then shows that the identity map

$$CF_*(M; H; \Lambda_{\mathbb{k}}) \to CM_*(M; H; \Lambda_{\mathbb{k}})$$

is a chain map and is hence a quasi-isomorphism.

However, this is only an intuitive argument and we are lack of a statement that H be regular. Since this is a special choice we have no way to run genericity arguments, and the resolution is to consider instead the ordinary equation

$$\frac{\partial u}{\partial s} = \nabla H(u)$$

computing gradient trajectories but with one extra variable, to prove that it's Fredholm and regular for generic choice of (H, J), and to compare this operator with our Floer operator. We will omit all the proof here and refer the reader to [AD14, Chapter 10]. $\square$

11 Problem. An alternative approach, due to Piunikin-Salamon-Schwarz [PSS96], is to construct directly a chain map

$$CF_*(M; \mathcal{F}; \Lambda_{\mathbb{k}}) \to CM_*(M; H; \Lambda_{\mathbb{k}})$$

for some Floer datum $\mathcal{F}$ and some Morse-Smale function H, by counting holomorphic "spines", which are unions of a pseudo-holomorphic plane and a gradient trajectory of H attached to it. Show that this map is a quasi-isomorphism of chain complexes, and moreover intertwines the pair-of-pants product on Floer cohomology with the quantum product on Morse cohomology, and therefore establishes the equivalence between Floer homology and quantum cohomology.

12 Problem. An alternative proof to this Hamiltonian invariance, which also has close relationship to algebraic K-theory, is to understand all the possible points in this family of Lagrangian submanifolds where they fail to intersect transversely. This is Floer's original proof, where we were able to provide a classification of all the possible situations. Then we can figure out how chain complexes change for these "elementary moves". This bifurcation analysis is related to Whitehead torsions of Floer complexes. State this result of Sullivan [Sul02] and explain the proof.

(5c) Lagrangian Floer homology. We now turn to the construction of Floer homology for Lagrangian intersections, which is the key ingredient in the proof of Theorem 5.4.

(5c-i) The action functional and Floer's equation. For two Lagrangian submanifolds $L, K \subseteq M$ intersecting transversely, we consider the path space

$$\mathcal{P}(L, K) = \{x \in C^\infty([0,1], M) | x(0) \in L, x(1) \in K\},$$

and in order to have a well-defined action functional, for fixed $p \in L \cap K$, we look at caps of the form

$$\Gamma(s,t)\colon [0,1]\times[0,1] \to M, \quad \Gamma(s,0)\in L, \quad \Gamma(s,1)\in K, \quad \Gamma(0,t)=p, \quad \Gamma(1,t)=x(t),$$

where Γ_1 and Γ_2 are homotopic iff there exists a homotopy Γ_τ of caps connecting Γ_1 and Γ_2. We then look at the abelian cover of $\mathcal{P}(L,K)$ given by

$$\widetilde{\mathcal{P}}(L,K) = \{(x,[\Gamma]) | \Gamma \text{ is a cap of } x\},$$

so that the action functional of $(x,[\Gamma])$ is given by

$$\mathcal{A}(x,[\Gamma]) = \int_{[0,1]\times[0,1]} \Gamma^*\omega.$$

This action functional defines a differentiable function on the infinite-dimensional Fréchet manifold $\widetilde{\mathcal{P}}(L,K)$, and the gradient vector field is given by

$$\nabla\mathcal{A} = J\frac{d\gamma}{\partial t},$$

so that the corresponding Floer's equation is given by

$$\begin{cases} \dfrac{\partial u}{\partial s} + J\dfrac{\partial u}{\partial t} = 0 \\ \lim\limits_{s\to+\infty} u(s,t) = p_+ \\ \lim\limits_{s\to-\infty} u(s,t) = p_- \\ u(s,0)\in L, u(s,1)\in K \end{cases} \tag{6}$$

Note that $\dfrac{\partial u}{\partial s} + J\dfrac{\partial u}{\partial t} = \bar{\partial}_J u$, so the Floer's equation for Lagrangian intersections is just the non-linear Cauchy-Riemann equation that we have discussed before, with different domains (a strip with boundary conditions and boundary asymptotics).

The energy of a solution $u\colon \mathbb{R}\times[0,1]\to M$ to Floer's equation is given by

$$E(u) = \int_{\mathbb{R}\times[0,1]} |du|^2 ds\, dt = 2\int_{\mathbb{R}\times[0,1]} u^*\omega,$$

and similar arguments as in the Hamiltonian case show that $E(u) < \infty$.

(5c-ii) Grading. As the choice of almost complex structures should not be essential, we consider one-parameter family of almost complex structures $J_t \in \mathcal{J}_\omega(M)$, $0 \le t \le 1$, and look at the moduli space of trajectories

$$\widehat{\mathcal{M}}(p_-,[\Gamma_-],p_+,[\Gamma_+];J_t,A) = \left\{ u\colon \mathbb{R}\times[0,1]\to M \,\middle|\, \begin{array}{c} \partial_s u + J_t\partial_t u = 0 \\ (6),\ [\Gamma_-\#u\#\Gamma_+] = A \end{array} \right\}.$$

Here $u\colon [0,1]\times\mathbb{R}\to M$ pulls back a symplectic vector bundle $u^*TM \to [0,1]\times\mathbb{R}$ whose restriction to the boundaries gives a pair of paths of Lagrangian subspaces $u(0,-)\colon \mathbb{R}\to \mathcal{L}Gr(T_{p_-}M)$ and $u(1,-)\colon \mathbb{R}\to \mathcal{L}Gr(T_{p_-}M)$ which compactifies to a pair of closed paths using the asymptotic condition. The index $\mu(u)$ is then defined to be the **relative Maslov index** of the two paths $L_u(0,-)$ and $L_u(1,-)$, which is the sum of the difference

$$\mu(u) = \sum_s \operatorname{sgn}\left(\Gamma(L_u'(1,s), L_u(0,s); s) - \Gamma(L_u'(0,s), L_u(1,s); s)\right).$$

This provides grading differences between intersection points of L and K, but in general the grading for Lagrangian submanifolds are more subtle. We follow here the framework provided by Seidel [Sei00] following an idea of Kontsevich. Given a compact symplectic manifold (M,ω), we can equip it with a fibre bundle

$$\mathcal{L}Gr(M) \to M$$

whose fibre at $p \in M$ is the Lagrangian Grassmannian $\mathcal{L}Gr(T_pM)$ of the tangent space T_pM.

36 Exercise. Show that $H^1(\mathcal{L}Gr(V);\mathbb{Z}) \cong \mathbb{Z}$ for any symplectic vector space (V,ω).

Proof. First note from universal coefficient theorem [Hat02] that we have a short exact sequence

$$0 \to \hom^1(H_0(U(n);\mathbb{Z})) \to H^1(U(n);\mathbb{Z}) \to \hom(H_1(U(n);\mathrm{Z}),\mathbb{Z}) \to 0$$

of $\mathbb{Z}$-modules, and since $U(n)$ is path-connected for all n, we know that $H^1(U(n);\mathbb{Z}) \cong \hom(H_1(U(n);\mathbb{Z}),\mathbb{Z})$. There is a Hurewicz map $\pi_1(U(n),1) \to H_1(U(n);\mathbb{Z})$ making $H_1(U(n);\mathbb{Z})$ the abelianization of $\pi_1(U(n),1)$.

Note that $U(n)$ naturally acts on the space $\mathbb{S}(V)$ of the sphere of unit vectors in the complex symplectic vector space (V,ω,J), which is surjective and a stabilizer subgroup of a given unit vector $v \in \mathbb{S}(V)$ is isomorphic to $U(n-1)$. Therefore we have the following long exact sequence of homotopy groups

$$\to \pi_2(\mathbb{S}(V)) \to \pi_1(U(n-1),1) \to \pi_1(U(n),1) \to \pi_1(\mathbb{S}(V),v) \to \pi_0(U(n-1)) = 0,$$

where if $n \geq 2$ we have $\pi_1(\mathbb{S}(V)) = \pi_2(\mathbb{S}(V)) = 0$, and hence $\pi_1(U(n),1) \cong \pi_1(U(n-1),1) \cong \pi_1(U(1),1) = \mathbb{Z}$ as $U(1) = \mathbb{S}^1$. $\square$

We call the corresponding fibrewise generating cohomology class $\mu_M \in H^1(\mathcal{L}\,\mathrm{Gr}(M);\mathbb{Z})$ a **Maslov class** on M. We want to ask whether for each choice of $N \in \mathbb{Z}$, there exists a $\mathbb{Z}/N\mathbb{Z}$-cover $\mathcal{L}^N\,\mathrm{Gr}(M)$ of $\mathcal{L}\,\mathrm{Gr}(M)$ so that fibres are $\mathcal{L}^N\,\mathrm{Gr}(T_pM)$, which is a $\mathbb{Z}/N\mathbb{Z}$-cover of $\mathcal{L}\,\mathrm{Gr}(T_pM)$ corresponding to the Maslov class $\mu_{T_pM} \in H^1(\mathcal{L}\,\mathrm{Gr}(T_pM);\mathbb{Z}/N\mathbb{Z})$. When $N=\infty$, we customly regard $\mathcal{L}^\infty\,\mathrm{Gr}(T_pM)$ as the $\mathbb{Z}$-cover over $\mathcal{L}\,\mathrm{Gr}(T_pM)$ corresponding to the Maslov class $\mu_{T_pM} \in H^1(\mathcal{L}\,\mathrm{Gr}(T_pM);\mathbb{Z})$.

5.20 Proposition. Given a symplectic manifold (M,ω), there exists a Maslov cover $\mathcal{L}^N\,\mathrm{Gr}(M)$ if and only if $2c_1(M) = 0$ in $H^2(M;\mathbb{Z}/N\mathbb{Z})$.

We will prove this fact in the next section.

5.21 Definition. A symplectic manifold (M,ω) is called **Calabi-Yau** if $2c_1(M)=0$ in $H^2(M;\mathbb{Z})$.

Suppose that $2c_1(M) = 0$ in $H^2(M;\mathbb{Z}/N\mathbb{Z})$ so that the N-th Maslov covering $\mathcal{L}^N\,\mathrm{Gr}(M)$ exists. A Lagrangian submanifold $L \subseteq M$ induces a natural lift $L \to \mathcal{L}\,\mathrm{Gr}(M)$ mapping each $p \in L$ to $T_pL \in \mathcal{L}\,\mathrm{Gr}(T_pM)$.

5.22 DEFINITION. A Lagrangian submanifold $i\colon L \hookrightarrow M$ is **gradable** if the lift $L \to \mathcal{L}Gr(M)$ admits a further lift to $\tilde{i}\colon L \to \widetilde{\mathcal{L}Gr}(M)$, and a choice of such lift is called a **grading** of L.

37 EXERCISE. Show that L is gradable iff the pull-back $\mu_L = i^*\mu_M \in H^1(L;\mathbb{Z})$ of the Maslov class μ_M vanishes.

When $\mu_L \neq 0$ we are still able to give a **relative grading** of a Lagrangian submanifold L, by considering instead the $\mathbb{Z}/N\mathbb{Z}$-cover of $\mathcal{L}\,\mathrm{Gr}(M)$ so that $N\mu_L = 0$.

5.23 DEFINITION. We call

$$N_L = \min\{|\mu_L(\gamma)|\colon \gamma \in \pi_1(L,*) \text{ and } \mu(\gamma) \neq 0\}$$

the **minimal Maslov number** of L.

(5c-iii) Regularity. The linearized operator $D_{u,J}$ for the Lagrangian Floer equation is exactly the same as in Section 3 with boundary conditions along the Lagrangian submanifolds.

5.24 THEOREM. Let (M,ω) be a compact symplectic manifold, then the following holds.

(i) The linearized operator

$$Ds_{(u,J_t)}\colon T_uW^{1,p}(\mathbb{R}\times[0,1],M;p_-,p_+;A)\times T_{J_t}\left([0,1]\times\mathcal{J}^{\ell}_{\omega}(M)\right) \to L^p(\mathbb{R}\times[0,1],u^*TM)$$

is split surjective.

(ii) For every $J_t \in \mathcal{J}_\omega(M)$ ω-compatible and every $u \in \mathcal{M}(p_-,[\Gamma_-],p_+,[\Gamma_+];J_t,A)$, the linearized operator D_{u,J_t} is Fredholm of index $\mu(u)$.

(iii) The projection $d\pi\colon T_uW^{1,p}(\mathbb{R}\times[0,1],M;p_-,p_+;A)\times T_{J_t}([0,1]\times\mathcal{J}^{\ell}_{\omega}(M)) \to T_{J_t}([0,1]\times\mathcal{J}^{\ell}_{\omega}(M))$ is Fredholm.

(5c-iv) Compactness. For intersections of Lagrangian submanifolds L and K we have the following compactness result:

5.25 THEOREM. Let $\{u_n\}$ be a sequence of J_t-holomorphic strips in $\widehat{\mathcal{M}}(p_-,[\Gamma_-],p_+,[\Gamma_+];J_t,A)$, then in a possible limit the following situation will arise:

- Bubbles or trees of bubbles appear at some of the interior points in $\mathbb{R}\times[0,1]$, just as what happens in the Uhlenbeck-Gromov compactness theorem.
- The trajectory $\mathbb{R}\times[0,1]$ breaks into several pieces, each of which connects a pair of intersection points with decreasing relative Maslov indices in order.
- A disk bubble appears at one of the boundary components $\mathbb{R}\times\{0\}$ or $\mathbb{R}\times\{1\}$, which is a J_t-holomorphic disk with boundary on L or K with exactly one boundary puncture.

The last situation is essential in that a disk bubble will drop the dimension of the moduli space by $\mu(\beta)$, which is possible to be 1 even if we assume monotonicity, and hence will essentially involve in the compactification of moduli space of J-holomorphic strips of dimension 1. This case is ruled out by the assumption that $\pi_2(M,L)=0$ and $\pi_2(M,K)=0$. Note that if K is hamiltonian isotopic to L, then these two assumptions are equivalent.

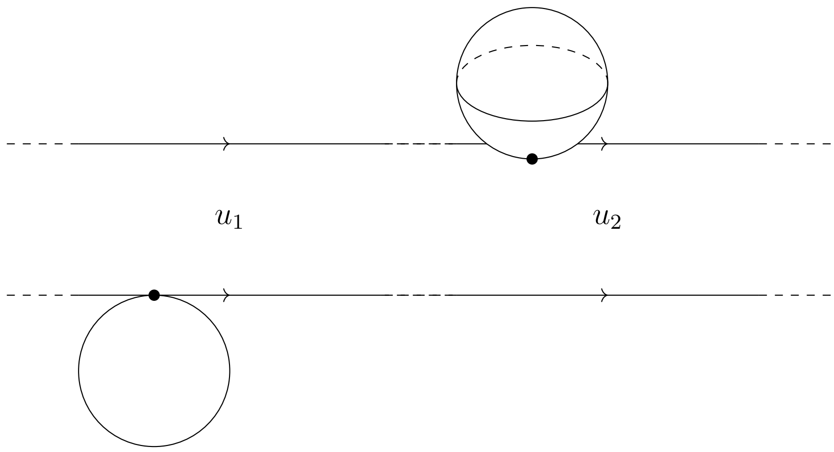

Figure 7: Possible breakings of J-holomorphic strips.

(5c-v) Gluing. Analogous to the Hamiltonian case, we need to understand how compactified strata are attached to the main stratum.

5.26 THEOREM. Suppose $\pi_2(M, L) = 0$ and $\pi_2(M, K) = 0$, then for generic choice of J_t, the moduli space of J_t-holomorphic strips

$$\mathcal{M}(p_-, [\Gamma_-], p_+, [\Gamma_+]; J_t, A)$$

of dimension ≤ 1 can be compactified into a smooth manifold with boundary consisting of broken trajectories

$$\mathcal{M}(p_-, [\Gamma_-], p_0, [\Gamma_0]; J_t, A) \times \mathcal{M}(p_0, [\Gamma_0], p_+, [\Gamma_+]; J_t, A),$$

where $\mu(p_+, [\Gamma_+]) - \mu(p_0, [\Gamma_0]) = 1$ and $\mu(p_0, [\Gamma_0]) - \mu(p_-, [\Gamma_-]) = 1$.

(5c-vi) The Floer complex. For a pair of Lagrangian submanifolds L and K satisfying $\pi_2(M, L) = 0$ and $\pi_2(M, K) = 0$, we can define the **Lagrangian Floer complex**

$$CF_*(L, K; J_t; \mathbb{Z}/2\mathbb{Z}) = \bigoplus_{p \in L \cap K} \Lambda_{\mathbb{Z}/2\mathbb{Z}} \langle p \rangle [-\mu(p)]$$

to be the $\mathbb{Z}$-graded $\mathbb{Z}/2\mathbb{Z}$-vector space generated by intersection points $p \in L \cap K$, where differentials are given by

$$\partial(p) = \sum_{\substack{q \in L \cap K \\ \mu(q) = \mu(p) - 1}} \sum_{A \in H_2(M, L; \mathbb{Z})} \# \overline{\mathcal{M}}(p, [\Gamma], q, [\Gamma']; J_t, A) T^{\omega(A)} \cdot q.$$

Again $\partial^2 = 0$, so we have the **Lagrangian Floer homology**

$$HF_*(L, K; J_t; \Lambda_{\mathbb{Z}/2\mathbb{Z}}) = H_*(CF_*(L, K; J_t; \Lambda_{\mathbb{Z}/2\mathbb{Z}}), \partial).$$

(5c-vii) Hamiltonian invariance. The Lagrangian Floer homology $HF_*(L, K; J_t; \Lambda_{\mathbb{Z}/2\mathbb{Z}})$ is also independent of the choice of the family of almost complex structures J_t. A similar argument as in the Hamiltonian case gives the following result, whose proof we omit.

5.27 PROPOSITION. Let $\{\phi_t\}$ be a Hamiltonian diffeomorphism of M, then there is a quasi-isomorphism

$$CF_*(L, K; J_t; \Lambda_{\mathbb{Z}/2\mathbb{Z}}) \to CF_*(L, \phi_1(K); J_t; \Lambda_{\mathbb{Z}/2\mathbb{Z}}).$$

In particular, we define the self-Floer homology of L to be

$$CF_*(L, L; J_t; \Lambda_{\mathbb{Z}/2\mathbb{Z}}) \coloneqq CF_*(L, \phi(L); J_t; \Lambda_{\mathbb{Z}/2\mathbb{Z}})$$

for some choice of hamiltonian diffeomorphism ϕ such that $\phi(L)$ is transverse to L.

(5c-viii) Floer homology on cotangent bundles and proof of Arnold's conjecture. To solve the Lagrangian version of Arnold conjecture, we only need to focus on the Floer complex $CF_*(L, \phi(L); J_t; \Lambda_{\mathbb{Z}/2\mathbb{Z}})$ for some hamiltonian diffeomorphism ϕ. Note by Weinstein's neighbourhood theorem 10 we can identify an open subset of L symplectically with an open subset of $L \hookrightarrow T^*L$ in the cotangent bundle of L, and for a chosen Morse function $f\colon L \to \mathbb{R}$, the Lagrangian isotopy $L_t = \Gamma(tdf) \subseteq T^*L$ provides a Hamiltonian isotopy sending L to $L_1 = \Gamma(df)$, which intersects transversely with L, and the intersection points are exactly critical points of f. Our goal is to identify $CF_*(L, L_1)$ homologically with the Morse complex $CM_*(L; f)$, so that the proof follows.

Clearly $\{L_t\}$ is the image under L of the Hamiltonian $H = \rho \cdot f \circ \pi$ where $\pi\colon T^*L \to L$ is the projection and $\rho\colon T^*L \to \mathbb{R}$ is a smooth function which is 1 near L and zero away from an open neighbourhood of L. Let ϕ_H^t be the flow of H, and we consider strips

$$u(s, t) = \phi_H^t(\gamma(s))$$

for curves $\gamma\colon \mathbb{R} \to L$ on L. Let $J \in \mathcal{J}_\omega(M)$ be any ω-compatible almost complex structure on M whose restriction to an open neighbourhood of L sends TL to T^*L, and we set $J_t = (\phi_H^t)^*J = d\phi_H^t \circ J \circ (d\phi_H^t)^{-1}$ to be a one-parameter family of ω-compatbile almost complex structures. Then we have

5.28 LEMMA. If γ is a gradient trajectory for f, i.e. $\dot\gamma(t) = \nabla f(\gamma(t))$, then $\bar\partial_{J_t} u = 0$.

Proof. By definition,

$$\frac{\partial u}{\partial s} = d\phi_H^t(\dot\gamma(s)) = d\phi_H^t(\nabla f(\gamma(s))),$$

while

$$J_t \frac{\partial u}{\partial t} = J_t \circ d\phi_H^t\left(X_H(\gamma(s))\right) = d\phi_H^t \circ J X_H(\gamma(s)) = -d\phi_H^t(\nabla f(\gamma(s))),$$

and therefore we have $\partial_s u + J_t \partial_t u = 0$. □

Conversely, suppose that $u\colon \mathbb{R} \times [0,1] \to M$ is a J_t-holomorphic strip with boundary on L and L_1, we can consider the modification

$$\tilde u(s, t) = \phi_H^{-t}(u(s, t)),$$

and we want $\tilde u(s,t)$ to be constant in t, i.e. $\dfrac{\partial \tilde u}{\partial t} = 0$. Let's compute that

$$\bar\partial_J \tilde u = \frac{\partial \tilde u}{\partial s} + J\frac{\partial \tilde u}{\partial t} = d\phi_H^{-t}\frac{\partial u}{\partial s} + J d\phi_H^{-t}\left(\frac{\partial u}{\partial t} - X_H(u)\right) = d\phi_H^{-t}\left(\frac{\partial u}{\partial s} + J_t\left(\frac{\partial u}{\partial t} - X_H(u)\right)\right),$$

so $\bar\partial_{J_t} u = 0$ implies that $\bar\partial_J \tilde u = -J d\phi_H^{-t} X_H(u) = -J X_H(\tilde u) = \nabla H(\tilde u)$.

5.29 Proposition. When $\pi_2(M, L) = 0$, all pseudo-holomorphic strips $u\colon \mathbb{R} \times [0,1] \to M$ with respect to $\{J_t\}$ are supported in the zero section L.

This Proposition allows us to reduce our discussion completely inside the cotangent bundle T^*L.

Proof. From assumptions we have $\omega(u) = 0$ for any $u\colon (\mathbb{D}^2, \mathbb{S}^1) \to (M, L)$. Note that $\tilde{u}$ is exactly a disk with boundary on L, so we have

$$0 = \omega(\tilde{u}) = \int_{\mathbb{R}\times[0,1]} \tilde{u}^*\omega = \int_{\mathbb{R}\times[0,1]} \left|\frac{\partial \tilde{u}}{\partial t}\right|^2 ds\wedge dt + \int_{\mathbb{R}\times[0,1]} \left\langle \frac{\partial \tilde{u}}{\partial s}, \nabla H \right\rangle ds\wedge dt + \int_{\mathbb{R}\times[0,1]} |X_{H_t}|^2\, ds\wedge dt.$$

On the other hand, we can compute the **geometric energy** of $\tilde{u}$ as

$$E(\tilde{u}|\mathbb{R}\times[0,1]) = \frac{1}{2}\int_{\mathbb{R}\times[0,1]} |du - X_H \otimes dt|^2 ds\wedge dt = \omega(\tilde{u}) + \int_{\mathbb{R}\times[0,1]} \left\langle \frac{\partial \tilde{u}}{\partial s}, \nabla H \right\rangle ds\wedge dt.$$

Note that $\omega(\tilde{u}) = 0$, so that

$$\int_{\mathbb{R}\times[0,1]} \left\langle \frac{\partial \tilde{u}}{\partial s}, \nabla H \right\rangle ds\wedge dt \geq 0.$$

Therefore we have

$$0 \leq \int_{\mathbb{R}\times[0,1]} \left|\frac{\partial \tilde{u}}{\partial t}\right|^2 ds\wedge dt = -\int_{\mathbb{R}\times[0,1]} \left\langle \frac{\partial \tilde{u}}{\partial s}, \nabla H \right\rangle ds\wedge dt - \int_{\mathbb{R}\times\mathbb{S}^1} |X_{H_t}|^2\, ds\wedge dt \leq 0,$$

so we conclude that $\dfrac{\partial \tilde{u}}{\partial t} \equiv 0$, and hence $\tilde{u}(\mathbb{R}\times[0,1]) \subseteq L$. □

Proof of Theorem 5.4. Proposition 5.29 implies that all pseudo-holomorphic strips $\tilde{u}$ have image on the zero section with equation

$$\frac{\partial \tilde{u}}{\partial s} = \nabla H(\tilde{u}),$$

which is exactly the gradient trajectory equation for f on L. Therefore $\tilde{u}$ corresponds to a gradient trajectory γ of f, and we hence obtain a bijective correspondence between the moduli space of pseudo-holomorphic strips $\mathcal{M}(p, [\Gamma], q, [\Gamma']; \{J_t\}, A)$ and the moduli space of gradient trajectories $\mathcal{M}(p, q; f)$. □

13 Problem. A slightly more general situation is when M is monotone and $L \subseteq M$ is also **monotone**, i.e. we can find a positive constant $\tau > 0$ so that for all disks $u\colon (\mathbb{D}^2, \mathbb{S}^1) \to (M, L)$, we have $\omega(u) = \tau\mu_L(u)$ where μ_L is the Maslov index on L. One special case is when $M = \mathbb{C}P^n$ with standard symplectic structure ω_{FS} and $L = \mathbb{R}P^n$ is the real locus of M. Show that for any Hamiltonian diffeomorphism ϕ of M such that $\phi(L) \cap L$ transversely, we have

$$\#\left(L \cap \phi(L)\right) \geq n + 1 = \sum_{i=0}^{n} \dim H_i(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}).$$

Reference: [Oh93b].

14 Problem. Another situation that is similar is when $M = \mathbb{C}P^n$ and

$$L = T^n = \left(\mathbb{S}^1\left(\frac{1}{\sqrt{n+1}}\right) \times \cdots \times \mathbb{S}^1\left(\frac{1}{\sqrt{n+1}}\right)\right) / \mathbb{S}^1 \hookrightarrow \mathbb{S}^{2n+1}/\mathbb{S}^1 = \mathbb{C}P^n$$

is the famously called **Clifford torus**. Show that no Hamiltonian diffeomorphism ϕ of M can separate L from itself. This is called **non-displaceability** of Lagrangian submanifolds, which can be obviously detected by Floer cohomology.

As an application, show that given any pairs of orthonormal bases $\{v_1, \cdots, v_N\}$ and $\{w_1, \cdots, w_N\}$ of a unitary vector space (V, ω, J) of dimension N, there exists a vector $x \in V$ such that

$$\langle x, v_i \rangle \langle v_i, x \rangle = \langle x, w_i \rangle \langle w_i, x \rangle = 1/N$$

for all $1 \le i \le N$.

Reference: [Oh93a, Lis26].

15 Problem. Show that for any monotone Lagrangian submanifold L of $\mathbb{C}^n$, we must have

$$1 \le \min\{\mu_L(u) | u \in \pi_2(M, L),\ \mu_L(u) > 0\} \le n.$$

Reference: [Oh96].

16 Problem. For Lagrangian Floer theory we also have a similar result to Piunikhin-Salamon-Schwarz. For a monotone Lagrangian submanifold $L \subseteq M$ in a monotone symplectic manifold, we can define the **Lagrangian quantum homology** $QH_*(L; \Lambda_{\mathbb{Z}/2})$ whose underlying module is $C_*(L; \mathbb{Z}/2)$ together with "quantized" product structure, counting the so-called **pearly trajectories**. Show that there is an isomorphism

$$HF_*(L, L; \Lambda_{\mathbb{Z}/2}) \cong QH_*(L; \Lambda_{\mathbb{Z}/2}).$$

Reference: [BC09].

6. Revisit Poincaré-Birkhoff

With Floer homology we should be convinced that Proposition 4.4 is completely resolved, by considering compactly supported Hamiltonian flow on $\mathbb{D}^*\mathbb{S}^1$ and computing Floer homology for this Hamiltonian flow. However, for symplectic manifolds with boundaries the Floer theory does not trivially work, due to the problem of the possibilities that a cylinder $u\colon \mathbb{R}\times\mathbb{S}^1 \to M$ asymptotic to Hamiltonian orbits of a compactly supported Hamiltonian flow can possibly "touch the boundary in its interior", i.e. there might be $(s,\theta)\in\mathbb{R}\times\mathbb{S}^1$ where $u(s,\theta)\in\partial M$. This is crucial when we want to prove the compactness result of Floer's cylinders, while it can be ensured by a variant of the so-called **maximum principle** for holomorphic curves in non-compact complex manifolds. We will address this issue in the first lecture.

Note that we still need to treat with fixed points of symplectomorphisms in $\pi_0\operatorname{Symp}_c(\mathbb{D}^*\mathbb{S}^1)$ and $\operatorname{Symp}_\infty(\mathbb{D}^*\mathbb{S}^1)$. In particular, $\pi_0\operatorname{Symp}_c(\mathbb{D}^*\mathbb{S}^1)$ is known to be generated by Dehn twists $\tau\colon \mathbb{D}^*\mathbb{S}^1\to\mathbb{D}^*\mathbb{S}^1$ which has have not fixed points away from its interior. Floer's original machinery only cares about symplectomorphisms isotopic to identity and has no implication for non-isotopic ones, while with inspiration from gauge theory, the subsequent work of Dostoglou-Salamon [DS93] established a version of Floer homology for arbitrary symplectomorphisms, and it was used by Seidel [Sei96, Sei97a, Sei99] to establish a significant amount of geometric applications. This is now called **fixed point Floer homology** which we will investigate in the second lecture and show that Dehn twists and their deformations must have at least two fixed points.

The final lecture will be devoted to the discussion of the effect of non-compact perturbations of compactly supported Hamiltonian diffeomorphisms. Unlike compactly supported ones, the continuation map for non-compact Hamiltonian homotopies can be defined only when the perturbation is "positive", due to the same maximum principle problem. Therefore we don't get a cohomology theory that only depends on the symplectic topology of $\mathbb{D}^*\mathbb{S}^1$ naively, while we can take limits with respect to the continuation map in order to get a Hamiltonian perturbation-invariant homology theory, which we call the **symplectic homology**. This was originally proposed by Floer and Hofer [FH94] while its importance was only realized after the famous work of Viterbo [Vit99] who applied symplectic cohomology theory to resolve Weinstein's conjecture for $\mathbb{C}^n$ and compute symplectic cohomology of cotangent bundles, leading to the famous **Viterbo isomorphism** relating symplectic geometry of cotangent bundles to string topology. We will investigate Viterbo's work in the next section.

(6a) Maximum principle. Instead of considering moduli space of cylinders in a compact symplectic manifold, it's more convenient to look at its completion into a non-compact symplectic manifold with cylindrical ends.

6.1 Definition. Let (M,λ) be a Liouville domain. The **completion** of M is the symplectic manifold

$$\widehat{M} = M\cup_{\partial M}(\partial M\times[1,+\infty)),$$

where $\partial M\times[1,+\infty)$ is equipped with the symplectic form $\hat\omega = d(r\lambda|_{\partial M})$ where r is the coordinate on $[1,+\infty)$. We call symplectic manifolds of this form **Liouville manifolds**.

Since M is a Liouville domain, the restriction $\lambda|_{\partial M}$ of λ to the boundary ∂M satisfies $\lambda\wedge(d\lambda)^{n-1}|_{\partial M}>0$, and the restriction of $d\lambda$ to the hyperplane distribution

$$\xi = \ker\lambda$$

defines a symplectic structure on $d\lambda$, so that the system of equations

$$\begin{cases} \lambda(R) = 1 \\ R \lrcorner d\lambda = 0 \end{cases}$$

defines a unique vector field R on ∂M, which we call the **Reeb vector field** associated to the 1-form λ. The Reeb vector field R induces a vector field on the cylindrical end $\partial M \times [1, +\infty)$, which we still denote by R.

6.2 DEFINITION. A $d\lambda$-compatible almost complex structure J on $\widehat{M}$ is **convex** if $JZ = -e^f R$ in the cylindrical end, where Z is the Liouville vector field of M and $f\colon \partial M \to \mathbb{R}$ is a smooth function.

Fix $\tau \in \mathbb{R}$ that is not a period of any 1-periodic orbit of R, we choose a Hamiltonian function $H_t^\tau\colon \widehat{M} \to \mathbb{R}$ such that in the cylindrical end, we have

$$H_t^\tau(x, r) = \tau r$$

when r is sufficiently large.

38 EXERCISE. Show that $X_{H^\tau} = \tau R$ in the cylindrical end.

The curves we are considering are smooth maps $u\colon \mathbb{R} \times \mathbb{S}^1 \to \widehat{M}$ satisfying the Floer equation

$$\frac{\partial u}{\partial s} + J\left(\frac{\partial u}{\partial t} - X_{H^\tau}(u)\right) = 0$$

with limits as $s \to \pm\infty$ being non-degenerate 1-periodic orbits of H_t^τ.

6.3 PROPOSITION (Maximum Principle). Let $u\colon \mathbb{R} \times \mathbb{S}^1 \to \widehat{M}$ be a solution to the Floer equation with respect to H^τ and a convex almost complex structure J. Then the image of u cannot intersect with the closed subset $\widehat{M}_\infty = \{(x, r) \in [0, +\infty) \mid H_t^\tau(x, r) = \tau r\}$ that contains an interior point.

Proof. We consider the **geometric energy** of u

$$\begin{aligned} E_{H^\tau}(u|\Sigma) &= \frac{1}{2}\int_\Sigma |du - X_{H^\tau} \otimes dt|^2 ds \wedge dt \\ &= \frac{1}{2}\int_\Sigma \omega\left(\frac{\partial u}{\partial s}, J_t \frac{\partial u}{\partial s}\right) + \omega\left(\frac{\partial u}{\partial t} - X_{H_t^\tau}, J_t\left(\frac{\partial u}{\partial t} - X_{H_t^\tau}\right)\right), \end{aligned}$$

and let Σ be the intersection of the image of u with $\overline{\widehat{M}_\infty}$, so that as $\omega = d\lambda$ is exact, by Stokes' theorem and the fact that u satisfies Floer's equation we have

$$E_{H_t^\tau}(u|\Sigma) = \int_{\partial\Sigma} u^*\lambda - \int_\Sigma u^* dH_t^\tau \wedge dt.$$

Note that in Σ we have $H_t^\tau = \tau r$, so that $dH_t^\tau = \tau dr$ and hence

$$\int_\Sigma u^* dH_t^\tau \wedge dt = \tau \int_\Sigma u^* dr \wedge dt = \tau \int_{\partial\Sigma} (r \circ u) dt = \tau \int_{\partial\Sigma} \lambda(R \circ u \otimes dt).$$

On the other hand, since J is convex, we have $\lambda = -e^f J^* dr$ in the cylindrical end, so that

$$\begin{aligned} E_{H_t^\tau}(u|\Sigma) &= \int_{\partial\Sigma} u^*\lambda - \tau\lambda(R \circ u \otimes dt) = \int_{\partial\Sigma} \lambda(du - \tau R \otimes dt) \\ &= -\int_{\partial\Sigma} \lambda \circ J_t(du - \tau R \circ u \otimes dt) \circ j = -\int_{\partial\Sigma} e^f dr(j^* du - \tau(R \circ u) \otimes ds) \\ &= -\int_{\partial\Sigma} e^f d(r \circ u) \circ j. \end{aligned}$$

Note that for any tangent vector $\xi \in T\partial\Sigma$ we must have $d(r \circ u) \circ (j\xi) \geq 0$ as $j\xi$ is inward pointing along Σ and $r \circ u$ achieves global minimum at $\partial\Sigma$. Therefore we conclude that

$$0 \leq E_{H_t^\tau}(u|\Sigma) = -\int_{\partial\Sigma} e^f d(r \circ u) \circ j \leq 0,$$

and hence Σ must be a measure zero set, which means it has to be at most a point. Elliptiness implies that the set of such points must be discrete. □

6.4 Corollary. Gromov compactness for Floer cylinders in $\widehat{M}$ for Hamiltonian functions with compact support holds, i.e. any sequence of Floer cylinders with uniformly bounded energy has subsequence that converges to only broken trjaectories.

Proof. Convergence is ensured by the maximum principle 6.3, and sphere bubbles will not appear as ω is exact. □

Furthermore, we have

6.5 Proposition. Suppose that $\{H_{t,s}^{\tau_s}\}_{t,s}$ is a one-parameter family of Hamiltonian functions with the given properties, then maximum principle for strips u satisfying Floer's equation

$$\frac{\partial u}{\partial s} + J_{t,s}\left(\frac{\partial u}{\partial t} - X_{H_{t,s}^{\tau_s}}(u)\right) = 0$$

holds when $\partial_s H_{t,s}^{\tau_s} \geq 0$ for all s, t in the cylindrical end.

Proof. In this case, the geometric energy is given by

$$E_{H_{t,s}^{\tau_s}}(u|\Sigma) = -\int_{\partial\Sigma} e^f d(r \circ u) \circ j - \int_\Sigma u^* \partial_s H_{t,s}^{\tau_s} ds \wedge dt$$

where Σ is the part of the strip in the cylindrical end. For maximum principle to hold, we therefore need the inequality

$$\int_\Sigma u^* \partial_s H_{t,s}^{\tau_s} ds \wedge dt \geq -\int_{\partial\Sigma} e^f d(r \circ u) \circ j,$$

which clearly holds when $\partial_s H_{t,s}^{\tau_s} \geq 0$. □

6.6 Corollary. The continuation map

$$c_{H_{s,t}} \colon CF_*(\widehat{M}; H_{t,0}^{\tau_s}, J_{t,0}; \Bbbk) \to CF_*(\widehat{M}; H_{t,1}^{\tau_1}, J_{t,1}; \Bbbk)$$

for a homotopy $\{H_{t,s}^{\tau_s}\}_{t,s}$ can be defined when $\partial_s H_{t,s}^{\tau_s} \geq 0$ in the cylindrical end, and it is independent of the choice of such homotopy.

In particular, with fixed slope τ smaller than all possible Reeb orbits it does not matter in the construction of the Floer homology $CF_*(H_t^\tau, J_t; \Bbbk)$ the behaviour of H_t^τ in the interior, and hence we can suppose that H_t^τ is C^2-small in the interior, so that only 1-periodic orbits of H_t^τ are constant orbits. We then have

6.7 Theorem. Let (M, λ) be a Liouville domain, then for any Hamiltonian diffeomorphism $\phi^t\colon M \to M$ that is time-independent, is C^2-small in the interior of M, and equals $\phi^t_{\tau r}$ in the cylindrical end for $r \gg 0$ sufficiently large and some τ smaller than all possible Reeb orbits, we have quasi-isomoprhism of chain complexes

$$CF^*(M; H_t^\tau, J_t; \Bbbk) \cong CM^*(M; H_t^\tau; \Bbbk).$$

Therefore we obtain the inquality

$$\#\operatorname{Fix}(\phi^1) \geq \sum_{i=0}^{2n} \dim H_i(M; \Bbbk).$$

Proof. The only non-triviality is about the Morse homology on non-compact symplectic manifolds. Note that the gradient flow of H^τ is always non-zero outside M, so that M deformation retracts via the reverse gradient flow to a compact CW complex whose CW homology is isomorphic to $H_*(M; \Bbbk)$, and hence we have $H_*(M; \Bbbk) \cong HM_*(M; H^\tau; \Bbbk)$. □

6.8 Corollary. Let ϕ be a Hamiltonian diffeomorphism of $\mathbb{D}^*\mathbb{S}^1$ that equals $\phi^1_{\tau r}$ on the boundary $\partial\mathbb{D}^*\mathbb{S}^1$ for some τ smaller than all possible Reeb orbits, then we have

$$\#\operatorname{Fix}(\phi) \geq 2.$$

Proof. Note that $H_*(T^*\mathbb{S}^1; \Bbbk) \cong H_*(S^1; \Bbbk) \cong \Bbbk^{\oplus 2}$. □

(6b) Fixed point Floer homology. For symplectomorphisms not isotopic to identity, the classical construction of Floer does not apply, though we have an alternative approach which was originally constructed by Dostoglou-Salamon [DS93] with motivations from gauge theory. The construction follows the formulation of Topological Field Theory, where we associate a circle with decoration $(\mathbb{S}^1, \phi)$ to a chain complex $CF^*(\phi)$, the **fixed-point Floer chain complex**, and to Riemann surfaces Σ with boundary components $\partial_-\Sigma = \sqcup_i (\mathbb{S}^1, \phi_i^-)$, $\partial_+\Sigma = \sqcup_j (\mathbb{S}^1, \psi_j^+)$ we associate a chain map

$$CF^*(\phi_1^+) \otimes \cdots \otimes CF^*(\phi_k^+) \to CF^*(\psi_1^-) \otimes \cdots \otimes CF^*(\psi_l^-).$$

In this section, we will only establish the chain complex $CF^*(\phi)$ and their differentials $d\colon CF^*(\phi) \to CF^{*+1}(\phi)$, and avoid discussion about its algebraic structure.

Let (M, λ) be a Liouville domain with completion $\widehat{M}$, and let $\phi \in \operatorname{Symp}(M)$ be an exact symplectomorphism. We define the **mapping circle** of ϕ to be the quotient space

$$\mathbb{S}_\phi(M) = \frac{M \times [0,1]}{(x,1) \sim (\phi(x), 0)},$$

which is naturally a symplectic fibre bundle over $\mathbb{S}^1$ with monodromy ϕ. The fibre bundle structure $\pi\colon \mathbb{S}_\phi(M) \to \mathbb{S}^1$ induces a symplectic hyperplane distribution

$$\xi_\phi \coloneqq \ker(d\pi),$$

with symplectic structure induced from that of M via the projection $M \times [0,1] \to M$ and the fact that ϕ preserves the symplectic structure. Write $d\theta$ to be the closed 1-form pull-back via π.

6.9 DEFINITION. The **Reeb vector field** on $\mathbb{S}_\phi(M)$ is a vector field R on $\mathbb{S}_\phi(M)$ such that $d\theta(R) = 1$ and $R \lrcorner \omega_\phi = 0$ where $\omega_\phi = (M \times [0,1] \to M)^*\omega$ is the pull-back 2-form on $\mathbb{S}_\phi(M)$.

> 39 EXERCISE. Show that the Reeb vector field R on $\mathbb{S}_\phi(M)$ is unique.

Proof. As ω_ϕ is non-degenerate on ξ_ϕ, by choosing a connection on $\mathbb{S}_\phi(M)$ we have a splitting $T\mathbb{S}_\phi(M) = \xi_\phi \oplus L$ for some line distribution L, and under this splitting we can write $R = R_v + R_h$ where $R_v \in \xi_\phi$ and $R_h \in L$. $R \lrcorner \omega_\phi = 0$ implies that $R_v = 0$, and $d\theta(R) = 1$ uniquely determines R_h, so that R is uniquely determined by the given equations. □

We would also like to perturb the Reeb vector field R to achieve some non-degeneracy. An **admissible Hamiltonian** is an $\mathbb{S}^1$-family of smooth functions $\{H_\theta \colon M \to \mathbb{R}\}_{\theta \in \mathbb{S}^1}$ such that $H_\theta(r, x) = \tau_\theta r$ for some constant τ_θ near the boundary of M. We can then define the **hamiltonian vector field** X_{H_θ} of H_θ by the equation

$$d\theta(X_{H_\theta}) = H_\theta, \ \ X_{H_\theta} \lrcorner \omega_\phi = 0.$$

As in usual Hamiltonian Floer theory, we consider the **twisted loop space**

$$\mathcal{L}_\phi(\widehat{M}) \coloneqq \left\{ \gamma \colon \mathbb{R} \to \widehat{M} \;\middle|\; \gamma(t+1) = \phi(\gamma(t)) \right\},$$

with **action functional** defined by

$$\mathcal{A}(\gamma) = -\int_0^1 \gamma^*\lambda - \int_0^1 H_\theta(\gamma(t))dt - F_\theta(\gamma(1)),$$

where $\phi^*\lambda - \lambda = dF_\theta$. Critical points of $\mathcal{A}$ corresponds exactly to twisted loops γ that satisfy the **Hamiltonian equation**

$$\mathcal{P}(\phi, H_\theta) = \{\dot{\gamma} = X_{H_\theta}(\gamma)\}.$$

We say a twisted Hamiltonian loop γ is **non-degenerate** if

$$\det(d\phi \circ d\phi_H^1 - \mathrm{Id}) \neq 0$$

at $T_{\gamma(1)}\widehat{M}$, and a twisted Hamiltonian $\{H_\theta\}$ non-degenerate if all its twisted Hamiltonian loops are non-degenerate.

In order to construct Floer cochain, we also pick a smooth 1-parameter family of ω-compatible almost complex structure $\{J_t\} \in \mathcal{J}_\phi(\widehat{M})$ that are convex in the cylindrical end and ϕ-periodic in the sense that $\phi^* J_{t+1} = J_t$ for all $t \in \mathbb{R}$. We call the pair (H_t, J_t) a **Floer datum** for ϕ.

The distribution $\xi_\phi \subseteq T(\mathbb{S}_\phi(M))$ defines a symplectic subbundle, so that with the choice of complex structure J_t we are able to regard ξ_ϕ as a complex vector bundle over $\mathbb{S}^1$.

> 40 EXERCISE. Show that any complex vector bundle $E \to \mathbb{S}^1$ over $\mathbb{S}^1$ is trivial.

Proof. We pick a CW structure on $\mathbb{S}^1$ as follows: vertices $\{0, 1/2\}$ where we identify $\mathbb{S}^1 = [0,1]/\{0,1\}$; edges $(0, 1/2)$ and $(1/2, 1)$. To get a trivialization of E over $\mathbb{S}^1$, we can first identify $E|_{0,1/2} \cong \mathbb{C}^r$ arbitrarily, and then trivialize $E|_{[0,1/2]}$ and $E|_{[1/2,1]}$ so that restricting the identification $E|_{[a,b]} \cong [a,b] \times \mathbb{C}^r$ to $E|_0$ coincides with the chosen trivialization. We then need to construct a family of matrices in $GL(n, \mathbb{C})$ that connects the transition map at $1/2$ to the identity, which is possible as $GL(n, \mathbb{C})$ is connected. □

For a twisted hamiltonian loop $\gamma \colon \mathbb{R} \to \widehat{M}$, the pull-back $\gamma^* \xi_\phi$ carries a natural symplectic connection given by differential of the hamiltonian flow $d\phi^t_{H_t}$, and under a trivialization $\gamma^* \xi_\phi \cong \mathbb{S}^1 \times \mathbb{C}^n$ we can identify $d\phi^t_{H_t}$ with a family of symplectic matrices $\{S_t\}_{t \in \mathbb{S}^1}$, so that the Conley-Zehnder index $\mu_{CZ}(\gamma)$ of γ is defined. However, we still need to understand whether this index depends on the choice of trivialization or not.

In order to avoid all the possible choices, we consider the Lagrangian Grassmannian bundle $\mathcal{L}Gr(M)$ from section (5c-ii). Recall that the action of $\mathrm{Sp}(V)$ on $\mathcal{L}\,\mathrm{Gr}(V)$ is transitive with stabilizer subgroup $\mathrm{St}(V)$, so that we can regard $\mathcal{L}\,\mathrm{Gr}(M)$ as the associated bundle of the principal $\mathrm{Sp}(2n)$-bundle

$$\mathrm{Sp}(M) \to M$$

under the standard action $\mathrm{Sp}(V) \curvearrowright \mathcal{L}\,\mathrm{Gr}(V)$. Recall that $\mathcal{L}\,\mathrm{Gr}(V)$ has a natural $\mathbb{Z}/N\mathbb{Z}$-covering $\mathcal{L}^N\,\mathrm{Gr}(V) \to \mathcal{L}\,\mathrm{Gr}(V)$ classified by the Maslov class $\mu_V \in H^1(\mathcal{L}\,\mathrm{Gr}(V); \mathbb{Z}/N\mathbb{Z}) \cong \mathbb{Z}/N\mathbb{Z}$, and we have an identification of $\mathcal{L}\,\mathrm{Gr}(V)$ with the homogeneous space $\mathrm{Sp}(V)/\mathrm{St}(V)$. Pick a compatible almost complex structure J, so that we can further reduce $\mathrm{Sp}(V)$ to $U(V, J)$, and we can find that

$$\mathrm{St}(V) \cap U(V, J) = O(n)$$

is the orthogonal group acting on the chosen Lagrangian subspace V_0.

41 Exercise. The determinant map $\det \colon \mathrm{Sp}(2n) \to \mathbb{S}^1$ induces an isomorphism of groups

$$\det{}^2_* \colon \pi_1(\mathcal{L}\,\mathrm{Gr}(V), L_0) \xrightarrow{\cong} \mathbb{S}^1$$

via squares of determinants. In particular, the action map $\mathrm{Sp}(V) \to \mathcal{L}\,\mathrm{Gr}(V)$ given by $S \mapsto S(L_0)$ induces a group homomorphism $\mathbb{Z} \cong \pi_1(\mathrm{Sp}(V), 1) \to \pi_1(\mathcal{L}\,\mathrm{Gr}(V), L_0) \cong \mathbb{Z}$ given by multiplication by 2.

Proof. The fibre bundle structure $O(n) \to U(n) \to \mathcal{L}\,\mathrm{Gr}(V)$ induces a long exact sequence of homotopy groups

$$\cdots \to \pi_1(O(n)) \to \pi_1(U(n)) \to \pi_1(\mathcal{L}\,\mathrm{Gr}(V)) \to \pi_0(O(n)) \to \pi_0(U(n)) \to \cdots$$

where $\pi_0(U(n)) = 1$ is trivial, $\pi_1(O(n)) = 1$ when $n = 1$, $\mathbb{Z}/2\mathbb{Z}$ when $n \geq 3$, and $\mathbb{Z}$ when $n = 2$, and $\pi_1(U(n)) \cong \mathbb{Z}$. In any case, we obtain an exact sequence

$$\cdots \to \pi_1(O(n)) \to \mathbb{Z} \to \pi_1(\mathcal{L}\,\mathrm{Gr}(V)) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

On the other hand, the determinant map $\det \colon U(n) \to \mathbb{S}^1$ induces a fibre bundle $SU(n) \times \mathbb{Z}/2\mathbb{Z} \to U(n) \xrightarrow{\det^2} \mathbb{S}^1$, so that we get a long exact sequence of homotopy groups

$$\cdots \to \pi_1(SU(n) \times \mathbb{Z}/2\mathbb{Z}) \to \pi_1(U(n)) \xrightarrow{\det_*} \pi_1(\mathbb{S}^1) \to \pi_0(SU(n) \times \mathbb{Z}/2\mathbb{Z}) \to \pi_0(U(n)) \to \cdots$$

Knowing that $SU(n)$ is simply connected, we have $\pi_0(SU(n)) = \pi_1(SU(n)) = 1$, and so we get another exact sequence

$$1 \to \pi_1(U(n)) \xrightarrow{\det^2} \pi_1(\mathbb{S}^1) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

From the comparison commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1(O(n),1) \arrow[r] & \pi_1(U(n),1) \arrow[r] \arrow[d, equal] & \pi_1(\mathcal{L}\,\mathrm{Gr}(V),L_0) \arrow[r] \arrow[d, "\det^2"] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[d, equal] & 1 \\
& 1 \arrow[r] & \pi_1(U(n),1) \arrow[r, "\det^2"] & \pi_1(\mathbb{S}^1,1) \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 1
\end{tikzcd}$$

we see that the middle morphism $\det^2\colon \pi_1(\mathcal{L}\,\mathrm{Gr}(V),L_0) \to \pi_1(\mathbb{S}^1,1) \cong \mathbb{Z}$ is an isomorphism, so that the quotient map $U(n) \to \mathcal{L}\,\mathrm{Gr}(V)$ is multiplication by 2 on the level of fundamental groups. $\square$

In particular, using $H^1(G;\mathbb{Z}/N\mathbb{Z}) \cong \hom(\pi_1(G),\mathbb{Z}/N\mathbb{Z})$ we see that the action pull-back $H^1(\mathcal{L}\,\mathrm{Gr}(V);\mathbb{Z}/N\mathbb{Z}) \to H^1(\mathrm{Sp}(V);\mathbb{Z}/N\mathbb{Z})$ maps the Maslov class μ_V to $2\delta_V$. Therefore the symplectic group $\mathrm{Sp}(V)$ has a $\mathbb{Z}/N\mathbb{Z}$-covering $\mathrm{Sp}^N(V) \to \mathrm{Sp}(V)$ corresponding to the generator $2\delta_V \in H^1(\mathrm{Sp}(V);\mathbb{Z}/N\mathbb{Z}) \cong \mathbb{Z}/N\mathbb{Z}$, which is also called the **Maslov covering** of $\mathrm{Sp}(V)$, so that the action of $\mathrm{Sp}(V)$ on $\mathcal{L}\,\mathrm{Gr}(V)$ lifts to an action of $\mathrm{Sp}^N(V)$ on $\mathcal{L}^N\,\mathrm{Gr}(V)$.

Now let's globalize the procedure. For a symplectic manifold (M,ω), we ask when $\mathrm{Sp}(M)$ admits the $\mathbb{Z}/N\mathbb{Z}$-covering $\mathrm{Sp}^N(M) \to \mathrm{Sp}(M)$ with fibre $\mathrm{Sp}^N(T_pM)$. Note that this question is equivalent to the existence of the N-th Maslov cover $\mathcal{L}^N\,\mathrm{Gr}(M) \to \mathcal{L}\,\mathrm{Gr}(M)$.

Proof of Proposition 5.20. The fibre bundle structure

$$1 \to \mathbb{Z}/N\mathbb{Z} \to \mathrm{Sp}^N(V) \to \mathrm{Sp}(V) \to 1$$

induces an exact sequence of nonabelian cohomology groups

$$0 \to H^1(M;\mathbb{Z}/N\mathbb{Z}) \to H^1(M;\mathrm{Sp}^N(V)) \to H^1(M;\mathrm{Sp}(V)) \xrightarrow{d} H^2(M;\mathbb{Z}/N\mathbb{Z}),$$

where $\mathrm{Sp}^N(M)$ is classified by the Čech cohomology classes in $H^1(M;\mathrm{Sp}^N(V))$ which restricts to a Čech cohomology class in $H^1(M;\mathrm{Sp}(V))$ that classifies $\mathrm{Sp}(M) \to M$, which are equivalent to $U(n)$-principal bundles and hence the map $H^1(M;\mathrm{Sp}(2n)) \to H^2(M;\mathbb{Z}/N\mathbb{Z})$ has image given by the first Chern class $c_1(M,\omega)$. Now the result follows. $\square$

42 Exercise. Consider the Liouville manifold $(T^*\mathbb{S}^1,\lambda_{\mathrm{tau}})$. Show that $c_1(T^*\mathbb{S}^1) = 0$.

Note that $2c_1(M) = 0$ always holds when $N = 2$, so that the Maslov covering $\mathcal{L}^2\,\mathrm{Gr}(M)$ of M always exists. Suppose that M admits an N-fold Maslov cover $\mathcal{L}^N\,\mathrm{Gr}(M)$, and let $\gamma\colon \mathbb{R} \to \mathbb{S}_\phi(M)$ be a non-degenerate twisted Hamiltonian loop, then we have $\phi^{-1}(\gamma(1)) = \gamma(0)$ and hence $d\phi^{-1} \circ d\phi^1_{H_t}|_{\gamma(0)}$ lifts to an element in $\mathrm{Sp}^N(T_{\gamma(0)}M)$. Consider a path $(\Phi,\widetilde{\Phi})\colon [0,1] \to \mathrm{Sp}^N(T_{\gamma(0)}M)$ connecting $(\mathrm{Id},\rho(k))$ for some Deck transform $\rho(k)$ to $d\phi^{-1}\circ d\phi^1_{H_t}|_{\gamma(0)}$, we then define its **absolute Conley-Zehnder index** to be

$$\mu_{CZ}(\gamma) = \mu_{CZ}(\Phi,\widetilde{\Phi}) = \mu_{CZ}(\Phi) + k \in \mathbb{Z}/N\mathbb{Z}.$$

It's not difficult to see that this index does not depend on all the choices.

Now we are able to define the **fixed point Floer chain complex** $CF^*(\phi; H_t, J_t; \Bbbk)$ associated to a given Floer datum to be the graded $\Bbbk$-module

$$CF_*(\phi; H_t, J_t; \Bbbk) = \bigoplus_{\gamma \in \mathcal{P}(\phi, H_t)} \Bbbk[\mu_{CZ}(\gamma) - n],$$

and we need to construct the differential $\partial\colon CF_*(\phi; H_t, J_t; \Bbbk) \to CF_{*-1}(\phi; H_t, J_t; \Bbbk)$ to finish the construction.

Given twisted Hamiltonian loops $\gamma_\pm \in \mathcal{P}(\phi, H_t)$, we count gradient flows connecting these two loops, which can be translated into a smooth map $u\colon \mathbb{R} \times \mathbb{R} \to \widehat{M}$ such that $u(s, t+1) = \phi(u(s,t))$ satisfying Floer's equation

$$\frac{\partial u}{\partial s} + J_t \left(\frac{\partial u}{\partial t} - X_{H_t}(u) \right) = 0$$

with limits as $s \to \pm\infty$ being $\gamma_\pm$. This is equivalent to twisted J_t-holomorphic sections of a fibre bundle $\mathbb{C}^*_\phi(M) \to \mathbb{C}^*$ over $\mathbb{C}^*$ with fibre M, and so the maximum principle 6.3 still holds. With standard elliptic theory we also know that the space of regular Floer data (H_t, J_t) is residual, where the dimension of the moduli space $\mathcal{M}(\gamma_-, \gamma_+; H_t, J_t)$ of such gradient trajectories is given by $\mu_{CZ}(\gamma_-) - \mu_{CZ}(\gamma_+) - 1$.

We define the differential $\partial\colon CF_*(\phi; H_t, J_t; \Bbbk) \to CF_{*-1}(\phi; H_t, J_t; \Bbbk)$ by

$$\partial(\gamma_-) = \sum_{\gamma_+ \in \mathcal{P}(\phi, H_t)} \#\mathcal{M}(\gamma_-, \gamma_+; H_t, J_t) \cdot \gamma_+,$$

where $\mathcal{M}(\gamma_-, \gamma_+; H_t, J_t)$ is the moduli space of Floer trajectories connecting γ_- and γ_+, and the count is zero if $\mu_{CZ}(\gamma_-) - \mu_{CZ}(\gamma_+) \neq 1$. Standard gluing argument implies that $\partial^2 = 0$, so that we have a well-defined chain complex $CF^*(\phi; H_t, J_t; \Bbbk)$, and we define the **fixed point Floer homology** $HF^*(\phi; H_t, J_t; \Bbbk)$ to be the homology of this chain complex.

6.10 PROPOSITION. Let ϕ^0 and ϕ^1 be exact symplectomorphisms on a Liouville manifold $(\widehat{M}, \lambda)$ that are equal to a Reeb flow ϕ_R^a for some $a \in \mathbb{R}$ in the cylindrical end, such that $\phi^1 = \phi_H^1 \circ \phi^0$ for some Hamiltonian diffeomorphism ϕ_H^1 supported in the interior of M, and let (H^0, J^0) and (H^1, J^1) be Floer data for ϕ^0 and ϕ^1 respectively such that $H^0 = H^1$ in a cylindrical end of $\widehat{M}$, then there is a canonical quasi-isomorphism

$$CF_*(\phi^0; H^0, J^0; \Bbbk) \cong CF_*(\phi^1; H^1, J^1; \Bbbk).$$

Therefore the fixed-point Floer homology $HF^*(\phi; H_t, J_t; \Bbbk)$ of an exact symplectomorphism ϕ does not depend on the Hamiltonian isotopy class of ϕ and the choice of Floer data.

The remaining of this section is devoted to a computation of the fixed-point Floer homology of the Dehn twist $\tau\colon T^*\mathbb{S}^1 \to T^*\mathbb{S}^1$ with a small Reeb perturbation near infinity. Recall that with the identification $\mathbb{D}^*\mathbb{S}^1 \cong \mathbb{S}^1 \times [-1, 1]$, the Dehn twist τ is given by

$$\tau(\theta, r) = (\theta + \pi r, r),$$

so that τ has no fixed points in the interior of $\mathbb{D}^*\mathbb{S}^1$ but is identity near the boundary $\partial\mathbb{D}^*\mathbb{S}^1 = \mathbb{S}^1 \times \mathbb{S}^0$. τ does not extend smoothly to a symplectomorphism of $T^*\mathbb{S}^1$ by identities, but we can perturb τ to be smooth as follows: let $\psi \in C^\infty(\mathbb{R})$ be a smooth function such that $\psi(r) = -1$ when $r \le -1$, $\psi(r) = 1$ when $r \ge 1$, and $\psi'(r) > 0$ for $r \in (-1, 1)$, and consider the perturbed Dehn twist

$$\tau_\psi(\theta, r) = (\theta + \pi\psi(r), r).$$

We will consider the fixed-point Floer homology of the composition $\phi_R^\varepsilon \circ \tau_\psi$, where ϕ_R^ε is the Reeb flow near the cylindrical end of $T^*\mathbb{S}^1$, for some small $\varepsilon > 0$ that is smaller than the smallest period of R. This Reeb flow ϕ_R^ε is the Hamiltonian time-1 flow of the function $H\colon T^*\mathbb{S}^1 \to \mathbb{R}$ such that $H = 0$ in a precompact subset of the interior of $\mathbb{D}^*\mathbb{S}^1$, depending only on r, monotone increasing, C^2-small, and such that $H(\theta, r) = \varepsilon|r|$ for $|r| \gg 0$.

6.11 Theorem. Under this situation, the fixed-point Floer homology of τ_ψ is given by

$$HF_*(\tau_\psi; H_t, J_t; \Bbbk) \cong H_*(\mathbb{S}^1; \Bbbk)^{\oplus 2},$$

as $\Bbbk$-modules.

As fixed-point Floer homology is Hamiltonian invariant, this provides us a complete estimation of fixed points of any exact symplectomorphism of $T^*\mathbb{S}^1$ that is close to Id near the cylindrical end.

6.12 Corollary. Any exact symplectomorphism of $T^*\mathbb{S}^1$ that equals ϕ_R^ε in the cylindrical end has at least 2 fixed points.

Proof of Theorem 6.11. We use the Hamiltonian flow ϕ_R^ε described above to perturb τ_ψ so that fixed points of τ_ψ are all non-degenerate. Let's first show the following:

6.13 Lemma. Suppose that $\phi\colon \mathbb{R}^n \supseteq U \to V \subseteq \mathbb{R}^n$ is a diffeomorphism from a precompact open set U to another open set V that is sufficiently close to Id: there exists a metric g on $\mathbb{R}^n$ so that

$$\sup_{x \in U} g(\phi(x), x)$$

is sufficiently small, then for each point $x \in U$, the germ ϕ_x is smoothly isotopic to Id_x.

Proof. The silly isotopy $\phi^t(p) := t\phi(p) + (1-t)p$ will define a smooth isotopy from Id to ϕ, and the norm bound ensures that for each t ϕ^t defines a diffeomorphism, after shrinking the open set U if necessary. □

The Dehn twist τ_ψ equals Id near the cylindrical end $\mathbb{S}^1 \times \{-1, 1\} \times [0, \infty)$, so that we can find $\delta > 0$ so that τ_ψ is sufficiently close to Id in the region $\mathbb{S}^1 \times \{-1, 1\} \times [-\delta, \infty)$. The restriction of τ_ψ to $\mathbb{S}^1 \times \{-1, 1\} \times (-\delta, \infty)$ is then smoothly isotopic to identity.

With this setting, with a similar argument as in Proposition 5.19 implies that for C^2-small Hamiltonian perturbation H with given behaviour at inifnity that gives the Reeb flow ϕ_R^ε, all twisted Hamiltonian loops of τ_ψ are constant loops, i.e. critical points of H. Therefore we have a chain-level identification $CF_*(\tau_\psi; H_t, J_t; \Bbbk) \cong CM_*(\mathbb{S}^1; \Bbbk)^{\oplus 2}$. To identify their differentials, we need to understand twisted strips $u\colon \mathbb{R} \times \mathbb{R} \to T^*\mathbb{S}^1$ that are asymptotic to these constant loops. Note that the symplectic area of any strip u is given by

$$\begin{aligned}\omega(u) &= \int_{\mathbb{R}\times[0,1]} u^*\omega = \int_{\mathbb{R}} \lambda\left(\frac{\partial u}{\partial t}\right)\bigg|_{t=1} - \lambda\left(\frac{\partial u}{\partial t}\right)\bigg|_{t=0} ds \\ &= \int_{\mathbb{R}} (\phi^*\lambda - \lambda)(\partial_t u) ds = F_\tau(\gamma_+) - F_\tau(\gamma_-) = 0,\end{aligned}$$

hence with a similar argument as in Proposition 5.29 we conclude that $\partial_t u \equiv 0$, which implies that u has to be a gradient flow line of ∇H, and hence we have

$$HF_*(\tau_\psi; H_t, J_t; \Bbbk) \cong H_*(\mathrm{Fix}(\tau_\psi); \Bbbk) \cong H_*(\mathbb{S}^1; \Bbbk)^{\oplus 2}. \qquad \square$$

17 Problem. Fixed point Floer homology can be applied to understand exotic symplectomorphisms, i.e. symplectomorphisms that are smooth but not symplectically isotopic to the identity. For example, Seidel in his thesis [Sei97b] showed that for every two-dimensional complete intersection other than $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ there exists such an exotic symplectomorphism. What is that and how he proved it? State the construction and sketch the ideas.

References: [Sei08].

(6c) Understanding wrappings. The final part of this section is to understand how the Floer cohomology changes when the perturbation near infinity is large.

More precisely, for a given Liouville domain (M, λ), we look at Hamiltonian functions $H_t^\tau \colon \widehat{M} \to \mathbb{R}$ such that

- H_t is C^2-small in the interior of the Liouville domain M;
- $H_t = \tau r$ in the cylindrical end $\partial M \times [0, \infty)_r$ for $r \gg 0$ and some positive constant $\tau > 0$ avoiding period of the Reeb flow on ∂M;
- $\partial_r H_t > 0$ in the cylindrical end.

We want to understand the behaviour of Floer cohomologies $HF_*(\phi^1_{H_t} \circ \psi; J_t; \Bbbk)$ where ψ is a given compactly supported symplectomorphism. The maximal principle 6.5 implies that if $H_t^{\tau_1} \preceq H_t^{\tau_2}$, then there is a natural continuation map

$$CF_*(\phi^1_{H_t^{\tau_1}} \circ \psi; J_t; \Bbbk) \to CF_*(\phi^1_{H_t^{\tau_2}} \circ \psi; J_t; \Bbbk),$$

which not necessarily induces an isomorphism on homology. However, we will show that $CF_*(\phi_{H_t^\tau} \circ \phi; J_t; \Bbbk)$ behaves like "persistent complex" with respect to the slope τ.

6.14 Proposition. Given τ and $\varepsilon > 0$ sufficiently small, the continuation map

$$c_\varepsilon \colon CF_*(\phi_{H_t^\tau} \circ \psi; J_t; \Bbbk) \to CF_*(\phi_{H_t^{\tau+\varepsilon}} \circ \psi; J_t; \Bbbk)$$

is a quasi-isomorphism.

Proof. Suppose that $\varepsilon > 0$ is such that there is no Reeb orbit of period in the interval $[\tau, \tau + \varepsilon]$, and in this situation we see that the generators of $CF_*(\phi_{H_t^\tau} \circ \psi; J_t; \Bbbk)$ and $CF_*(\phi_{H_t^{\tau+\varepsilon}} \circ \psi; J_t; \Bbbk)$ are the same, and so we expect the continuation map to be an identity map away from the interior of M.

To see this, recall that the energy of a continuation strip u is given by

$$E_{H_{t,s}^{\tau_s}}(u) = \mathcal{A}_{H_{t,s}^{\tau_+}}(\gamma_+) - \mathcal{A}_{H_{t,s}^{\tau_-}}(\gamma_-) - \int_{\mathbb{R} \times [0,1]} u^* \partial_s H_{t,s}^{\tau_s} ds \wedge dt,$$

so we must have the inequality

$$\mathcal{A}_{H_{t,s}^{\tau_+}}(\gamma_+) - \mathcal{A}_{H_{t,s}^{\tau_-}}(\gamma_-) \geq \int_{\mathbb{R} \times [0,1]} u^* \partial_s H_{t,s}^{\tau_s} ds \wedge dt \geq 0.$$

Therefore we must have $\mathcal{A}_{H^{\tau_+}_{t,s}}(\gamma_+) \geq \mathcal{A}_{H^{\tau_-}_{t,s}}(\gamma_-)$, and hence the image of the continuation map must be twisted Reeb orbits of increasing action.

Let's look at the particular case when γ_- and γ_+ are twisted Reeb orbits connected by a continuous family of twisted Reeb orbits $\{\gamma_+^\alpha\}$ of increasing slope, then we can consider the moduli space of Floer cylinders

$$\mathcal{M}(\gamma_-, \gamma_+^\alpha; J_t, H^\alpha_{s,t}) = \{(u,\alpha) | u \in \mathcal{M}(\gamma_-, \gamma_+^\alpha; J_t, H^\alpha_{s,t}), \alpha \in [0,1]\}$$

which is a smooth manifold with boundary for generic choices of Floer data and the path $\{H^\alpha_{s,t}\}$. Note that the possible boundary strata of $\mathcal{M}(\gamma_-, \gamma_+^\alpha; J_t, H^\alpha_{s,t})$ are given by $\alpha \in \{0,1\}$ or horizontal compactifications via Gromov-Uhlenbeck compactness, which we see from exactness of $\widehat{M}$ that this cannot happen. Therefore the signed count of $\mathcal{M}(\gamma_-, \gamma_+^1; J_t, H^1_{s,t})$ and $\mathcal{M}(\gamma_-, \gamma_+^0; J_t, H^0_{s,t})$ are the same, while when $\alpha = 0$ we have $H^0_{s,t} = H^\tau_{s,t}$ and so

$$E_{H^0_{t,s}}(u) = \mathcal{A}_{H^\tau_{t,s}}(\gamma_+^0) - \mathcal{A}_{H^\tau_{t,s}}(\gamma_-) = 0,$$

implying that u has to be a constant strip. Therefore we see that the continuation map must satisfy $c_\varepsilon(\gamma_-) = \gamma_+^1$ for all twisted Reeb orbits γ_-.

These twisted Hamiltonian orbits are actually canonically identified via homotopies of paths, and so we can identify the chain complexes $CF_*(\phi_{H^\tau_t} \circ \psi; J_t; \Bbbk) \cong CF_*(\phi_{H^{\tau+\varepsilon}_t} \circ \psi; J_t; \Bbbk)$, so that c_ε is represented by a matrix which is identity along the diagonal, with $\langle c_\varepsilon(\gamma_-), \gamma_+\rangle = 0$ when $\mathcal{A}(\gamma_-) \leq \mathcal{A}(\gamma_+)$ and $\gamma_- \neq \gamma_+$. This implies that c_ε is upper-triangular when restricted to twisted Reeb orbits. On the other hand, as compactly supported perturbation does not affect Floer chain complex, we can a priori assume that H_t is a constant C^2-small Morse function in the interior of M, so that c_ε is identity on interior critical points/twisted Hamiltonian loops, and therefore c_ε is an isomorphism of chain complexes. □

Therefore the continuation map c_ε fails to be an isomorphism only when $[\tau, \tau+\varepsilon]$ contains a period of the Reeb flow. In this case, the Hamiltonian Floer homology $HF_*(\phi_{H^\tau_t} \circ \psi; J_t; \Bbbk)$ does depend on the choice of the non-compactly supported Hamiltonian diffeomorphism, but we can form an inductive limit

$$SC_*(\psi; J_t; \Bbbk) = \operatorname*{hocolim}_{\tau \to \infty} CF_*(\phi_{H^\tau_t} \circ \psi; J_t; \Bbbk),$$

where the diagram of Floer chain complexes are provided by the continuation map c_{τ_s} for some family τ_s. This is called the **symplectic chain complex** of ψ, and its homology $SH_*(\psi; J_t; \Bbbk)$ is called the **symplectic homology** of ψ. This homology group is now independent of the choice of Hamiltonian perturbation and is hence an invariant of ψ. When $\psi = \mathrm{Id}$, we simply write $SH_*(\mathrm{Id}; \Bbbk) = SH_*(\widehat{M}; \Bbbk)$, which is an invariant of $\widehat{M}$.

(6d) Weinstein conjecture and symplectic homology. One application of symplectic homology is to show Weinstein's conjecture in certain cases. We will start with a brief introduction to this conjecture. Recall that a distribution $\xi \subseteq TY$ on a smooth manifold Y is called **integrable** if $[\xi, \xi] \subseteq \xi$, in which case they are tangent spaces to a family of smooth submanifolds of Y that exhaust Y. Any such distribution can be written locally as kernels of a tuple of differential 1-forms, i.e. $\xi|_U = \ker \alpha_1 \cap \cdots \cap \ker \alpha_k$ for some local coordinate chart U and some 1-forms $\alpha_1, \ldots, \alpha_k$ on U. In particular, if ξ is a hyperplane distribution, then locally $\xi = \ker \alpha$ for some 1-form α. The Frobenius theorem states that ξ is integrable if and only if locally for any defining 1-form α, $\alpha \wedge d\alpha = 0$.

Conversely, we can use the rank of $\alpha \wedge d\alpha$ to describe the non-integrability of ξ. ξ is said to be **maximally non-integrable** if $\alpha \wedge d\alpha$ has maximal possible rank, and when Y has odd dimension, this is equivalent to $\alpha \wedge (d\alpha)^n$ being a volume form on Y.

6.15 Definition. Let Y be a smooth manifold. A hyperplane distribution $\xi \subseteq TY$ is called a **contact structure** if ξ is maximally non-integrable. ξ is **co-orientable** if there is a global defining 1-form α such that $\xi = \ker \alpha$. In this case, α is called a **contact form** for ξ. We refer to (Y, ξ) as a **contact manifold**.

43 Exercise. Show that for a contact manifold (Y, ξ) and a contact 1-form α, $d\alpha|_\xi$ is a non-degenerate 2-form on ξ.

44 Exercise. Let (M, λ) be a Liouville domain, then $(\partial M, \lambda|_{\partial M})$ defines a contact manifold with the restriction of λ a contact 1-form.

Similar to mapping torus, we can define the **Reeb vector field** R associated to a contact form α to be the unique vector field on Y satisfying

$$\begin{cases} \alpha(R) = 1, \\ d\alpha(\cdot, R)|_\xi = 0. \end{cases}$$

A simple closed curve $\gamma\colon [0,T] \to Y$ is called a **Reeb orbit** if $\dot\gamma(t) = R(\gamma(t))$ for all $t \in [0,T]$. Note that when $Y = \partial M$ is a boundary of some Liouville manifold (M, λ), where the contact 1-form is given by $\alpha = \lambda|_{\partial M}$, then a Reeb orbit of α is the same as a Hamiltonian orbit of the Hamiltonian function $H\colon \widehat{M} \to \mathbb{R}$ in the cylindrical end, where $H = r^2$ near the cylindrical end.

6.16 Conjecture (Weinstein). Let (Y, ξ) be a closed contact manifold and α a contact 1-form on Y, then α has at least one Reeb orbit.

Suppose that (Y, α) admits a Liouville filling (M, λ), let's explain how symplectic homology helps us with Weinstein's conjecture. When we say (Y, α) admits a Liouville filling (M, λ), we mean M is a Liouville domain where $Y = \partial M$ is the boundary of M, and $\alpha = \lambda|_{\partial M}$ is the restriction of the Liouville 1-form. The symplectic chain complex $SC_*(\widehat{M}; \mathbb{k})$ is generated by Hamiltonian orbits of an increasing sequence of Hamiltonian functions $H_t^\tau\colon \widehat{M} \to \mathbb{R}$ whose Hamiltonian vector field coincides with some scalar multiples of the Reeb vector field in the cylindrical end, so that it roughly consists of Morse critical points of a Morse function on $\widehat{M}$ together with Reeb orbits of the contact boundary ∂M. By taking the slope of H_t^τ to be close to 0, we see a continuation map

$$H^*(M; \mathbb{k}) \cong HM_*(H_t^\tau; \mathbb{k}) \xrightarrow{c_{0,\infty}} SH_*(\widehat{M}; \mathbb{k}),$$

coming from composition of any $c_{\varepsilon,\tau}\colon HF_*(\phi_{H_t^\varepsilon}; \mathbb{k}) \to HF_*(\phi_{H_t^\tau}; \mathbb{k})$ with the colimit map $HF_*(\phi_{H_t^\tau}; \mathbb{k}) \to SH_*(\widehat{M}; \mathbb{k})$. If $c_{0,\infty}$ fails to be an isomorphism, then there must be non-trivial Reeb orbit of $\lambda|_{\partial M}$ that contributes to the failure of $c_{0,\infty}$ to be an isomorphism. We therefore get the following algebraic criterion for Weinstein conjecture:

6.17 Conjecture (Algebraic Weinstein Conjecture). Let (M, λ) be a Liouville domain, then the continuation map $c_{0,\infty}\colon H^*(M; \mathbb{k}) \to SH_*(\widehat{M}; \mathbb{k})$ is not an isomorphism.

Let's end this section with a proof of conjecture 6.17 in the case when Y is the boundary of $(\mathbb{C}^n, \lambda_{std})$, the symplectic vector space regarded as a Liouville manifold, where the Liouville 1-form is given by $\lambda_{std} = \frac{1}{2}\sum_{i=1}^{n}(x_i dy_i - y_i dx_i)$.

6.18 Remark. It does not really matter which Liouville 1-form we choose on $(\mathbb{C}^n, \omega_{std})$, as they're all **Liouville homotopic** to each other, i.e. there is a piecewise smooth family of Liouville 1-forms $\{\lambda_t\}$ that connects given Liouville 1-forms on $\mathbb{C}^n$, so that zeroes of λ_t all lie inside a given compact subset, and when such Liouville homotopy exists, their symplectic homologies are quasi-isomorphic, and hence the result applies to all convex contact hypersurfaces in $\mathbb{C}^n$.

6.19 Theorem (Floer-Hofer-Wysocki). $SH_*(\mathbb{C}^n; \Bbbk) = 0$, and hence Weinstein's conjecture holds for all convex contact hypersurfaces in $\mathbb{C}^n$.

Proof. Consider the Hamiltonian function $H^\tau(z) = \frac{\tau}{2}|z|^2 = \tau\sum_{i=1}^{n}|z_i|^2$, so we can compute that

$$X_{H^\tau}(z) = \tau\sum_{i=1}^{n}\frac{\partial}{\partial\theta_i},$$

where ∂_{θ_i} is the rotation vector field in the i-th complex coordinate. Integral curve of X_{H^τ} is then given by

$$z(t) = (z_1 e^{i\tau t}, \ldots, z_n e^{i\tau t}),$$

which forms an orbit only when $\tau \in 2\pi\mathbb{Z}$ or $z_1 = \cdots = z_n = 0$. Therefore for $\tau \in \mathbb{R}_{>0}\setminus\mathbb{Z}$, the Floer chain complex

$$CF_*(H^\tau; \Bbbk) \cong \Bbbk$$

consists of the constant orbit $z(t) \equiv 0$, where the family of symplectic matrices linearizing $\phi^t_{H^\tau}$ looks like

$$d\phi^t_{H^\tau}(0) = \begin{pmatrix} e^{i\tau t} & 0 & \cdots & 0 \\ 0 & e^{i\tau t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\tau t} \end{pmatrix},$$

so for $k \le \tau < k+1$, the Conley-Zehnder index of the constant orbit $z_\tau(t)$ is

$$\mu_{CZ}(z_\tau) = 2\sum_{i=1}^{n}\left\lfloor\frac{\tau}{2\pi}\right\rfloor + n = 2nk + n.$$

On the other hand, continuation maps have degree 0, and therefore in the homotopy colimit all such orbits should disappear, and we conclude that $SH_*(H^\tau; \Bbbk) = 0$. ☐

18 Problem. The symplectic homology of a Liouville domain (M, ω) also satisfies Künneth formula. Show that for Liouville domains (M_1, ω_1) and (M_2, ω_2), there is a quasi-isomorphism

$$SC_*(M_1 \times M_2; \Bbbk) \simeq SC_*(M_1; \Bbbk) \otimes SC_*(M_2; \Bbbk).$$

References: [Oan06].

19 PROBLEM. One can use symplectic homology to detect symplectomorphism types of Liouville manifolds, or more specifically **Stein manifolds**, which are holomorphic manifolds that form a specific class of Liouville manifolds. Consider the complex hypersurface $V = \{x^7+y^2+z^2+w^2 = 0\} \subseteq \mathbb{C}^4$ and a smooth point $p = (0, 0, 1, i) \in V$. Let H be the blow-up of $\mathbb{C}^4$ along p and let $X = H \setminus V^{prop}$ be the complement of the proper transform of V. Show that $SH_*(X) \neq 0$, so that X is diffeomorphic but not symplectomorphic to $\mathbb{C}^n$.

Actually there exists an infinite family of such examples in any dimension. What are they?

Reference: [McL09].

20 PROBLEM. Symplectic homology can also be used to detect the affineness of Stein manifolds. Consider cotangent bundles T^*M where M is closed. If M is a surface of genus $g \geq 2$, or a manifold with fundamental group that is the free product of at least three non-trivial groups, show that T^*M is not symplectomorphic to any smooth affine variety.

Reference: [McL12].

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